Part I

The theoretical frame

The Theoretical Frame and The Glossary

1. The current state of teaching/learning of algebra

1.1. Introduction

International research and literature regarding mathematical learning, and in particular algebraic propensity and its difficulties, – at diverse ages from junior levels through to university – have underlined a widespread traditional teaching method quandary. The reasons found are completely different:

- *Cognitive:* the hurdle towards generalizing and symbolic thinking is in 'itself' a difficult jump;
- *Psychological:* algebra intimidates students who are already frustrated by difficult arithmetic approaches;
- *Social:* families and more in general the environment consciously or unconsciously transmit behaviour traits which can be defined as being 'mathematic phobias'. As Cesare Cornoldi (Cornoldi, 1995) recalls, an idea which is often associated with mathematics is that mathematics applies *central* intellective abilities (an unsuccessful result seems more definite and irreversible here, than in other subjects) and it can be ascertained that the parents' attributes in this sense are passed on to their children (as many teachers hear repeat during parents' meetings);
- *Pedagogical:* students seem less and less inclined towards education and seem less motivated towards studying, especially when – as is the case with algebra – a higher step and effort is requested;
- *Didactical:* Italian secondary school teachers - traditionally secondary school (11-13 year olds) is where algebra is first taught - do not possess, generally speaking, a mathematical formation and therefore substantially only reproduce beliefs, scansions and stereotypes which were acquired when the teachers themselves were students and learnt algebra.

Of course, each of these reasons contains a fraction of truth which may interfere negatively with algebraic learning; our prospective has taken this into account, however we refer to a profoundly diverse starting point which we will now illustrate.

1.2. Future hypotheses regarding teaching and learning algebra

In December 2001 the 12th ICMI STUDY (International Committee for Mathematical Innovation) was held in Melbourne. The committee's choice of the theme title was highly significant: *The future of the Teaching and Learning of Algebra*. It is worthwhile reflecting upon some of the chief areas developed during this convention.

Algebra, as a language in itself of a *higher level* of mathematics, can be considered, by some students, as a *bridge* towards further studies, whereas for many others it is considered as a *wall*. From this starting point stem crucial questions: can algebra be made more accessible by modifying the quantity and difficulty of that which is taught? (many countries are making an effort in this direction, in the hopes of obtaining better results from students). Is algebra really useful to the *majority* of people? If this were the case *today*, would it continue to be so in the *future?*

An algebraic curriculum is completely different today from that of a few years ago, also due to various forms of technology which are completely modifying the concept of mathematical knowledge and *utility* and are inducing changes on what will be taught in the future. Over the last twenty years, a substantial area of research has taken form – area which would be the basis for the hypotheses of modernizing the curriculum.

Well then: why algebra? The principal answers represent three viewpoints: one, due to the fact that it supplies access to basic scientific and technological comprehension in a growing field of professions; two, it represents a *cultural heritage* and three, it contributes to the *formation* of citizens who possess instruments of objective and formative criticism.

For many it is a *barrier*, and this aspect, for its implications, must no be undermined: should algebra be taught to everybody? Which algebra? Is there an *acceptable minimum threshold* of algebraic knowledge? How can the algebraic curriculum be modified so it assumes a *value* for those who are studying it? Can we individualize significant contexts for students, in which algebra will take on a clear values and not an ambiguous ones ?

Over the past twenty years, research has focalized on a large number of possible approaches that increment the *meaning* of the algebraic *processes and objects*. One of the principal forms is: the *problem solving* (where emphasis is given to the analysis of problems and equations), the *functional* approach (the use of letters to indicate measurement and the formal coding of relations among measurements), the *generalization* approach (the use of expressions to represent geometric patterns, numerical sequences, ‘rules’).

A determining role is attributed to the *linguistic* approach and to research that affronts the didactical developments starting from the *concept of algebra as a language.* This role becomes even more significant if it is associated to *the hypothesis of an early start to algebraic education beginning from the didactical readings of the relations between arithmetic and algebra.* Research does demonstrate just how the students’ limited arithmetic experience becomes an obstacle when learning algebra. It is thought that an earlier approach can reduce this difficulty.

It is only recently that interest has been shown towards an early approach to algebra, thus there is not yet much documentation regarding this area. Question and answers are being formulated, such as:

- How *early* should *early algebra* be?
- What are the *advantages and disadvantages* of an anticipated start?

- How are the answers to these questions connected to *theory* of cognitive development and learning, and to the cultural and education t*raditions* of teaching algebra?
- Which algebra and algebraic thinking aspects should be part of an early algebraic education?
- What consequences would an early algebraic education have on *teachers and their formation*?

As we will see in the rest of this documentation, this final question summarizes one of the most crucial aspects connected to innovative didactical experiences found in the ArAl Project.

1.3. Links between arithmetic and algebra

The ArAl project is situated within that theoretic frame that assumes the denomination of *early algebra (*an early start to algebraic thinking): *in which it is thought that the principle cognitive obstacles are to be found in the pre-algebraic field, and that many of these spring up from unsuspected arithmetic contexts and they then become conceptual obstacles to the development of algebraic thinking*. Numerous recent studies in the field of international algebraic didactics, demonstrate how students lack appropriate arithmetic structures from which they can generalize, and moreover, how students lack the knowledge of arithmetic procedures and do not posses a conceptual base from which to build up their algebraic knowledge.

The didactical problems regarding elementary algebra can be identified at the construction level of:

(a) basic arithmetic knowledge

(b) algebraic knowledge

The first level (which corresponds roughly to the ages of between 6 to 12 year olds) does not give sufficient attention to the passage to algebra; the second level (traditionally around the age of 13) tends to concentrate excessively on the calculus processes. The result being that algebraic thinking is not constructed progressively as a thought tool and instrument *paralle*l to arithmetic, but s*uccessive* to arithmetic, thus above all its manipulative mechanisms and computational aspects are highlighted. Therefore algebra loses some of its essential characteristics: one, an appropriate language to describe reality and two, a potent reasoning and forecast instrument of codifying through formulas knowledge (or hypotheses) regarding phenomena (in our case elementary) and where new knowledge derives (by means of transformation consented by algebraic formalism) on the phenomena themselves knowledge (or hypotheses) regarding phenomena (in our case elementary) and where *new knowledge derives from* (by means of transformation consented by algebraic formalism) on the phenomena themselves.

Let us now follow a reverse path. We will begin by proposing some reflections about algebraic didactics in secondary schools (11-13 year olds) and then we will climb back up along the branches of an imaginary genealogy tree towards the 'arithmetic ancestors' of algebraic concepts

1.4. Potentially misleading models

Every secondary school mathematics teacher comes face to face with algebraic questions like: in the areas of themes, arguments, episodes, problems, logical passages, etc. when do students have more difficulty? Which are the most frequent errors? Which activities can influence/favour/be an obstacle to learning algebra? These are essential questions which permit a behaviour evaluation not only of students but of teachers too. They are a reference point for a more general question, that being: *'Which algebra do I teach?'*

As mentioned previously, traditional teaching is going through a crisis; further to the reasons listed initially, it is a belief that two probable causes of didactical and psychological nature are:

- investing too much time in technique exercises,
- the lack of recognizing psychological and cognitive barriers that impede the students' acceptance of the algebraic language.

When posed with questions: Why do so many students make mistakes when learning the formal algebraic language? Why are mistaken opinions and knowledge so resistant to change?, international research among 12-13 year olds responds that elementary algebraic notions are not necessarily difficult, however the defects can be found in the didactical practice which does not take sufficiently into consideration:

a) a widespread inadequacy of arithmetic comprehension,

b) linguistic difficulties connected to learning a formal language.

We shall discuss the linguistic difficulties later on, here we would like to point out some examples of how legitimate models relating to operations acquired in an arithmetic ambient may be mis-leading or inhibiting to the conceptual progress of an algebraic ambient:

- some research begins from the consideration that the model of multiplication as a repeated addition learnt at elementary school implies that *multiplying and multiplier* are *whole* numbers, for example: the student 'sees' '5 + 5 + 5 + 5' as '5 x 4', read as '5 repeated 4 times'. Later, however at an algebraic level, if the writing '3x' refers to that model, and therefore is interpreted as '3 repeated x times', many students lose track of the meaning in front of that '3' repeated '*how many* times?' due to the fact that students cannot 'see' the number of times. On the other hand, if the student is capable of interpreting '3x' as 'x + x + x', that is 'x repeated 3 times' then the passage of a not whole multiplier may form yet again a logical passage that is difficult to grasp: in ‘0,3x’ repeating x *for 0,3 times* is senseless because it does not have a comforting or concrete support.

- Even knowing from arithmetic that multiplication goads from commutative property, students often see multiplying and multiplier as things having a *different status*. For example, in the algebraic field, in ‘2y’ they see ‘2’ as a *different* entity from ‘y’. Also because, although they are able to grasp in ‘2y’ the commutative property and therefore the equivalence between ‘two times y’ and ‘y times 2’, if they write ‘y2’ the teacher will tell them they are wrong – thus consolidating their misconceptions in that which could be defined as a diverse ontology between the *number* and the *letter* (of course it is of fundamental importance that the concept of convention is studied in-depth).

1.5. Natural Language and Formal Language

Difficulties like these, within an arithmetic field, then influence that long chain of possible errors that students encounter when they face setting up an equation of a problematic situation. For example students :

- attempt (just as an *naive* translator would) to ‘literally’ translate the text;
- do not know or do not use the algebraic notation conventions;
- interpret number as adjectives, and letters as labels or as abbreviations;
- interpret an equation as a sequence of instructions, in which case the sign ‘=’ means ‘give place to’;
- do not know how to interpret the texts of ‘non-sequential translation’ problems, meaning problems in which the order of the terms used in the text is not satisfactory to their method of mathematical elaboration;
- do not clearly distinguish the sums and powers produced (ambiguity between additive and multiplicative structures);
- have confused ideas about ratio and difference.

It is believed that unconscious habits and cognitive process present within a natural language may create conflict with the procedures required from a formal language. For example: ‘y is three times bigger than z’ is *literally* translated erroneously as ‘y = 3 × + z’ (‘three times more than z’) or ‘y = 3 × > z’ (‘three times bigger than z’). In other words, it is presumed that without a complete awareness of arithmetic procedures and writings, students possess an impoverished conceptual base which impedes their future construction of algebraic knowledge.

However, it is opportune to underline that often, students’ errors and misconceptions are neither stupid nor lighthearted and they represent a result of reflection and reasonable attempts to attribute a meaning to mathematical expressions that would otherwise lack significance. Others could indicate reasoning one might define, as not as being wrong rather as being *interrupted* and could therefore represent the beginning of a potentially productive reasoning.

2. Cultural key elements of the ArAl project

2.1. A collective construction of meanings

The abovementioned consideration about the presence of a potentially productive reasoning brings us to what we have written previously about early learning algebra. Some youngsters – above all the younger ones at elementary school ages – are less conditioned by errors and stereotypes and express themselves more creatively and are more willing to amuse themselves. Thus within the class they can be lead to a collective construction of new *meanings* through the practice of reflections interpretation hypotheses, 'murky' language use, which are aspects often destined to remaining in the limbo of '*the unsaid'* thus creating errors and mis-conceptions which hinder the students' relationship with mathematics and more in general their relationship with school.

On a linguistic level, some of the major difficulties that younger students have to face with algebra, are represented by having to understand:

i) *why* a symbolic language has to be used;
ii) which *rules* does the symbolic language have to abide to;
iii) the difference between *solving* and *representing* a problem.

The prospective of initiating students to algebra as a language, within a continual backwards and forwarding use of arithmetic thinking, may favour the individualization of a more effective didactics with students aged between six and fourteen, as it is based on *negotiating* and thus on the *explicitness* of a didactical contract aimed at the solution of algebraic problems based on the principle *'first represent and then solve'*. This perspective (developed in depth further on) seems promising when facing one of the most important conceptual areas of algebra: the transformation of *representation* terms from the natural language in which they are formulated into the formal algebraic language translating the relations that they contain. In this way the search for the solution is transferred to the next phase.

Before facing the duality of represent/solve, one must concentrate on a fundamental point of the theoretical profile that the ArAl project refers to.

2.2. Algebraic babbling

We retain that there is a huge similarity between learning a natural language and learning an algebraic language; so, as to explain this point of view we have adopted the *babbling* metaphor.

When a child is learning a language he/she slowly approaches its meanings and rules and gradually develops these through imitation and use until school, when the child then learns to read and reflect upon the grammar and syntax aspects of the language. In traditional didactics of algebraic language one begins by firstly studying the *rules*, as if formal manipulation came before the comprehension of meanings. There is the tendency to teach algebraic syntax however at the same time its semantics are overlooked. Mental models of Algebraic thinking should instead be built up through what we call initial forms of *algebraic babbling*. Our hypotheses is that algebraic thinking and mental models of thought should begin from the first years of elementary school – years in which pupils begin to encounter arithmetic thinking, making it possible to teach them to *think about arithmetic in an algebraic way*. In other words, building up *progressively* in students algebraic thinking as an instrument and object of thought closely *interwoven with arithmetic.* Starting from its *meanings* and by means of constructing an environment that informally stimulates an autonomous elaboration of *algebraic babbling*, thus favouring the experimental approach to a new language in which the rules position themselves gradually, and within a didactical contract which tolerates initial 'promiscuous' syntactical moments.

2.3. Solve and Represent: product and process

These considerations lead us to a delicate area of construction on behalf of the students and their *ideas* about mathematics, ideas which participate in the formation of that which Schoenfeld refers to as the students' *epistemology*. Referring to what he/she is convinced about thus leading him/her retaining that the solution to a problem (a simple addition for elementary students or a more complex problem for an older student) is essentially – or exclusively -: *the search for a result.* This of course moves the students' concentration towards that which can produce said result, that being the *operation.* Solving problems basically means *calculating*. Students should be aided into thinking of how to distance him/herself from the worries of the result and consequently of the operations that will permit reaching that result. In this way, they should be aided into reaching a higher level of thinking: substituting the act of *calculating with 'looking at oneself' while calculating*. It is the passage from a cognitive level to a meta cognitive one in which one has to *interpret the structure of the problem.*

We can apply this reasoning to algebra. The perception, for example, of the *regularity* of a numeric succession, means discovering the *key to algebraic reading* of the problem in question. Identifying the structure consents the passage to algebraic writing, that being its *representation* in a formalized language. Thus algebra becomes a language not only to describe reality but one by which we may amplify its *comprehension.* This kind of process takes places very slowly, by progressive succession and through continual intersections and fractures among various levels of knowledge.

Traditional arithmetic didactics, however, tends instead to favour students' mental thought process in its immediate search of instruments (operations) for the individualization of the *solution* (the result), reminding one of a child's initial tendency to use the language to satisfy his/her *primary* needs (hunger, sleep, pleasure, …). A small yet significant example: probabely the latter behaviour is also induced by standard problems as:

> There are 13 crows perched on one branch; on another branch there are 6.
> *Calculate the total number of crows.*

With time pupils grasp that the verbal language has a much vaster scope of functions which are diversified in respect to initial solution and needs. Pupils learn to *describe* reality by penetrating the infinitive complexities and contradictions of its structures. At the same time getting to know themselves and therefore the purpose and functions of their own thought processes.

Something similar should also occur in arithmetic and algebra. The development of arithmetic thinking, characterized by operations on well known numbers, provokes the formation of stereotypes which are impossible to eradicate thus caging students within an obsessive search of a numeric result (*the primary need*) and blocking their explorations of diverse mental paths, which are infinitively more fruitful and stimulating for the formation of algebraic thinking in an embryonic form at this stage (*the interpretation and description of reality* using a mathematical language).

With this as an objective, the abovementioned problem should be re-formulated, thus consenting even very young to contemplate themselves and their calculations (thereby activating argumentative aspects which are far from being banal):

> There are 13 crows perched on one branch; on another branch there are 6.
> *Explain how to find the total number of crows and then calculate.*

The first example of this problem aims at the individualization of the *product,* that is the operations that consent one to *solve* the problem – whereas the second example of the problem aims at the individualization of the *process*, that being the writings which permit one to *represent* the manifestation of an articulate thought process.

In the first case the *diachronic* aspect prevails: the mental processes of calculation take place sequentially in time and the result emerges at the end of an *action*. In the second case the time dimension disappears: the author abstains from *doing* and poses his/her interest in the conceptual dimension of individualizing the *structure* that the alogorythm applied.

This is a basic concept to understanding the passage from an *arithmetic* way of thinking to an *algebraic* way of thinking. This is a very delicate step because it is linked to one of the most important aspects of the epistemological gap between arithmetic and algebra concerning the explicit and implicit contracts supporting the two procedures: whilst arithmetic requires an *immediate* search of a solution, on the contrary algebra *postpones* the search of a solution and begins with a formal trans-positioning from the dominion of a natural language to a specific system of *representation*.

In our opinion and as we have underlined previously, the perspective of algebra as a language can enhance the individualization of a more efficient didactics with students aged between seven and fourteen, due to the fact that it is based on *negotiating* and therefore *explicating* a didactical contract aimed at solving algebraic problems based on the principal *'first represent, then solve'*. A promising perspective when facing one of the most demanding and important areas of conceptual algebra: *the transposition in terms of natural language representation in which problems are formulated and described, into the formal algebraic language in which first the relations they contain are translated and then their solutions are found.*

We feel that profound changes are necessary within algebraic teaching spheres at lower secondary school, and that starting from elementary school it is opportune to *anticipate the approach to these problems. This can be done beginning from the individualization of didactical concepts which favour the passage from arithmetic thinking to algebraic thinking.*

Here are some examples of what we mean.

2.4. Examples

2.4.1 Diverse representations of a number

Among the infinitive representations of a number, the *canonical* representation is, for obvious reasons, the 'favourite'. Thinking of a number means, for anyone, thinking of its *cardinality*. However, this reasonable obviousness risks creating a stereotype and thus creating a barrier to the expansion of mathematical thinking in the passage from arithmetic to algebra.

The canonical representation is *opaque in meaning*, hence *saying little about itself* to the student. For example: the writing '12' suggests a generic 'number of things' and likewise the idea of 'even number'. Other representations – according to age – may amplify the field of information: '3×4 ' underlines the meaning of being a multiple of 3 and of 4; whereas '$2^2 \times 3$' means it is also a multiple of 2; and '$2 \times 2 \times 3$' leads to '2 x 6 ' and therefore to the multiple of 6; 36/3 or 60/5 meaning they are under multiples of other numbers and inserted under the root in the form of $\sqrt{2^2 \times 3}$ helps in the passage to$2\sqrt{3}$, and so on. One may affirm that each of the possible connotations of a number adds useful information which increments its meaning, just as words add meaning when describing people: Giancarlo, Alice's father, Cosetta's husband, the mathematics professor, the author of this paper, Francesco's friend, the Italian speaker and so on, all represent the same subject from different view points so they amplify knowledge in respect to the ‘canonical’ “Giancarlo Navarra”. Further on we will see a concrete example of how to get student used to conceiving as a ‘number’ not only ‘12’ but also ‘9 + 3’ or ‘$2^2 \times 3$’as an important passage towards the solution to various groups of problems and the comprehension of writings like ‘a + b’ o ‘x^2y’.

2.4.2 The equals sign

In arithmetic teaching at elementary school the equals sign essentially expresses the meaning of *directional operator*. For the student 4 + 6 = 10 means: I sum 4 and 6 to obtain 10. This is a dominant concept for the first seven, eight years of school during which the equals sign sign possesses a dominant connotation of *space and time*: it prepares the conclusion to a *story* which must be read from left to right (sequential operations take place) until one reaches its conclusion (and finally a result is obtained). Then, traditionally (in Italy) in the third year of secondary school (13-14 year olds), students meet algebra and the equals sign sign suddenly takes on a new and totally diverse meaning: it indicates the *equivalence between two quantities*. In a writing like ‘8 + x = 2x – 5’ it takes on a *relational* meaning and contains the idea of *symmetry between two writings*. Out of the blue (usually without being previously ‘informed’ of this enlargement of meaning), students have to be able to move around in a conceptual universe which is totally new and different, and within which it is necessary to go *beyond* the familiar connotation of space and time. However if the students’ understanding is ‘that the number after the equals sign sign is the *result*’ it is probable that the writing ‘11 = n’ means very little, even if the student can solve the first grade equation leading to it.

2.4.3 The properties of the operations

Generally, in normal mathematic didactics of primary and secondary school, the properties of the operations are relegated within a standard niche and hold little meaning. The commutative and associative properties are grasped through intuition and remain as stereotype synonyms of ‘change’ and ‘put together’. The hard part is made up of representing the distributive property, difficult to do at elementary school but frequently used in algebra in which, however, it is not often represented as such - rather as something completely new, (in Italy even its name is new : *extraction - or collection - of a common factor).*

We shall now look at just how these and other arithmetic and algebraic knots - which are so intensely interwoven - can be faced within the prospect of a linguistic approach to mathematic didactics.

2.5. Mathematics as a language

Previously we have mentioned the necessity to individualize the most productive didactical concepts, so as to favour the passage from arithmetic thinking to algebraic thinking; one of the foremost concepts is that linked to the relation between *natural* *language* and *mathematical language,* bearing in mind just how closely linked the capacity to correctly express a proposal in the *natural language* is and how to express and formulate it in the *algebraic* language is. One may affirm that the first difficulties faced when learning arithmetic and algebra are those connected to linguistic difficulties: organizing the speech, co-ordinating the phrases, describing objects and situations, providing definitions, recognition, following a reasoning and arguing through the solution of a problem. In other words, algebra should be favoured as a language that not only consents the description of reality but it also *amplifies its comprehension*. This process should take place very slowly, through a endless interwoven continuation and fracture between levels of mathematical knowledge.

2.6. Syntax and semantics

As in every language, even the mathematical language has its own *grammar rules*, that is, a set of conventions that permit one to correctly construct phrases (these may sometimes vary slightly: Italian didactics separates the co-ordinates of a point by using a comma, whereas in other countries a semi-colon is preferred so as not to create confusion when decimals are used. The language also possesses a *syntax*, that supplies the conditions – or rather the rules - to establish if a succession of linguistic elements is 'well formed' (for example , the following are incorrect syntax phrases "9 + + 6 = 15" or the classical chain of operations added one after the other like "5 + 3 = 8 : 2 = 4 + 16 = 20"). The language also possesses a *semantics*, that permits one to interpret symbols (within a correct syntax succession) and to then be able to establish if the expressions are true or false (for example the phrase "1 + 1 = 10" is true or false according to the base of calculation; it is false in the base 10 system yet true in the base 2 system).

Which of the two analysis – the syntax or semantic – which should come before the other? We can understand this by reflecting on the meaning of a sentence:

'Alice's fine'

It can be interpreted in two ways:

'Alice is fine'

'Alice's fine

The sentence leads to a diverse syntactical analysis:

Alice (subject) is (verb) fine (adjective)

Alice's (possessive case) fine (noun / traffic fine)

The point we would like to get across here is that the *syntax analysis necessarily follows the semantic analysis*, so the implications of this statement are extremely relevant.

In the prospective that we are considering, the *translation* of phrases from a natural language (graphic or iconic) into a mathematic language and vice versa, represents one of the most fertile territories within which one may develop reflections regarding the mathematical language. Translating in this sense means interpreting and representing a problematic situation by means of a formalized language or, on the contrary, recognizing within a symbolic writing set the situation it describes.

3. Methodological aspects

3.1.The ArAl project and teachers

What we have written so far synthesizes the frame which contains the ArAl project's activities. Important mathematical and linguistic aspects are dealt with here : relations between arithmetic and algebra, between syntax and semantic aspects of a language, between solving and representing a problematic situation, between natural and mathematical languages and between process and product, with the objective of individualizing a pathway for a curriculum that supports an anticipated approach to algebraic thinking, and therefore:

- constructing algebra as a language which possesses its own grammar rules and syntax (signs, conventions of writings, ect.,) favouring the translation operations from one language to another;
- privileging the comprehension of the meaning of the algebraic writings, thus avoiding that the students unconsciously manipulate the symbols (it is not part of the Project's objectives however, with elementary students, to build up instrumental abilities);
- favouring an approach to algebra, that, even at a simple level, contains all the characteristics of this discipline, in its widespread and modern meaning, by inserting activities that use algebra, not only as an analysis of procedures, but also as a language, a thinking instrument and a mathematical instrument to strengthen the resolution of problems and the individualization and confrontation of relations and structures.

However, up until this moment (July 2002), the ArAl project's principle users are teachers of elementary and secondary school (6-13 years old pupils) who, in general, do not have a mathematical university formation, as the majority come from areas of humanistic and pedagogical education (elementary teachers) and scientific education (secondary school teachers).

Hence, the project presents itself as an important occasion for teachers to reflect upon their *knowledge* (which, of course, conditions the choice of modalities through which teachers themselves then transmit their knowledge to students) and their c*ertainties* regarding mathematics – one could say regarding their *epistemology.* The situations laid down in each unit take place in stimulating didactical environments – which can often be difficult to handle – and require various competences and numerous delicate capabilities on behalf of teachers. In other words, teachers who intend facing innovative didactics must be prepared to encounter, as mentioned previously, his/her knowledge, competence and certainties together with a mix of methodological and organizational aspects. These are *not at all secondary aspects*, in fact they operatively support the actual *culture of change.*

3.2. Some significant aspects

3.2.1. The didactical contract

Constant check up on the clearness of the didactical contract in all of its phases, especially in elementary school. This means that the objective of the project is not that of supplying technical competence in advance (for example, through Unit 6 ‘‘From the Scales to the Equations’’ it does not intend teaching how to solve a first grade equation). The objective is to investigate which are the more adapt forms for building up mathematical concepts in students that will help them towards a gradual formation of algebraic thinking.

Students must be made to understand what is the essence of the didactical contract: that they are the *prime characters* in the collective construction of *algebraic babbling*. This means educating them gradually towards even complex forms of a new language by favouring their reflections on the differences and equivalences of mathematical writings and its meanings – a gradual discovery of the use of letters instead of numbers – the application of properties – the understanding of the different meanings of equals sign – the infinitive representations of a number, and so on. Naturally over the passage of time, this ingenuous discovery of rules and meanings must of course evolve into the students’ control of syntax and semantic rules, so as to reach the capacity of *communicating* through mathematical language in areas of problems, hypotheses and solutions, as for example happens in the activities of *Brioshi Project*, transversal to the all units in the ArAl project.

3.2.2. Discussions about mathematical themes

Activating a collective discussion about mathematical themes leads to privileging *meta-cognitive and meta-linguistic* aspects; students are led to reflect upon language, knowledge and processes (solving a problem and translating it into algebraic language). They also have to face hypotheses and their classmates’ proposals, to compare and classify translations, to evaluate their own beliefs and to apply responsible choices. Thus teachers must be aware of the *‘risks’* and of the particularities of this type of teaching method.

For example, the discussions regarding the various translations proposed by students about a sentence in natural language like: *‘Translate for Brioshi the phrase: From 15 take away 8’*. This means that the students themselves must be able to confront what they have written even if their translations are only slightly different and indicative of different conceptual contexts: the translation ‘15 – 8’ favours the transparency of the *process,* whereas ‘7’ is the search for a *result* and ‘15 – 8 = 7’ underlines *both aspects (see Unit 1: ‘Brioshi Project’*).

Being aware of the presence of two points of view is determining and permits one to grasp - and to help grasp – aspects linked to ‘doing mathematics’ that do not emerge in a traditional didactics, but which richens with important meanings the passage from an arithmetic environment to an algebraic one. Discussion helps to increment potential in *thinking arithmetic with an algebraic key,* and research has highlighted just how much *verbalization and argumentation* are fundamental vehicles for understanding.

3.2.3. Protocol interpretation

Building up competences for a refined interpretation and the successive classification of students’ proposals and protocols means being faced with an enormous variety of mathematical writings, which are often elaborated with a mixed and personalized use of language and symbols which have been put together more or less correctly. This behaviour is developed well if the teachers themselves stimulate creativity as well as reflection. When students realize they are the *producers of mathematical thinking* and are contributing to a collective construction of knowledge and languages, they express a huge variety of proposals, many of which are far from being banal and which, when put together, represent a common patrimony of all the class. This is the important point where the teacher needs the capacity to pick out (and to let students pick out) the paraphrases of a possibly correct sentence by selecting the wrong, ambiguous, bizarre, translations.

These are important activities as they help not only the students, but moreover the teacher, in understanding that every text in whatever mathematical context can be read and interpreted at different levels, even due to the organization and formulation of the text in a natural language. A refined analysis of written protocols and of students’ affirmations supplies the key to understanding in a punctual way their attempts, errors, misconceptions, stereotypes, difficulties in controlling meaning and to preparing modifications and corrections concerning teacher’s didactical engineering. Where possible, one should organize the collection of notes made during a class activity (students observations, teachers considerations, samples of significant answers) so as to back up a successive roundtable discussion about the activities which have taken place.

Let us conclude this section with a consideration made above all for elementary teachers who intend experimenting with some Units of the project: the activities that promote pre-algebraic thinking should not represent a ‘foreign body’ but should be positioned in relation to the mathematical curriculum with the objective of individualizing the modality for progressive integration. Thanks to our up to date experience as researchers and experimenters, this is a sensitive theme among teachers. On one hand it reflects the fear of having to ‘*make space for algebra’* within an already dense program, and on the other hand it reflects the opportunity of *self criticism regarding knowledge, arithmetic beliefs, contents, methods and strategies.*

4. Topics of the Units

Unit 1: Brioshi and the approach to algebraic code (7 – 12 year olds)

The Unit privileges the approach to *linguistic* aspects of mathematics. They are developed around an imaginary character, Brioshi, a Japanese student who can only communicate using the mathematical language, and who enjoys exchanging problems and solutions with classes of other nations. The units propose *translation* activities from natural language to arithmetic language and vice versa, starting from simple phrases like ‘From 4 take away 2’ and progressing to more complex activities like the ‘Game of the hidden number’ (‘To a hidden number add four to obtain ten’). This Unit demonstrates how the exchange of messages may begin by traditional tools (simulations, notes, faxes) until it reaches the ‘mathematical communication’ of two classes (contemporarily by means of a chat line set up using *MSN Messenger Service* software.

The approach to mathematical linguistic aspects is privileged and helps to reflect upon the relational aspects between elements of a problem of a mathematical writing (implicit when ‘representing’ a problem) and the procedural aspects (implicit when ‘solving’ a problem) and upon the symbolic language (syntax and semantic aspects and conventions – it is to be remembered that Brioshi only understands the mathematical language).

Let us pause briefly on this Unit as the activities it contains are transversal to all of the others. The messages, especially when used with younger students, can contain sentences written in Italian («Dear Brioshi, I want to see if you can …» or in Japanese (little tricks are necessary and Internet or a sinologist friend can lend a hand here, see examples in Unit 5). These sentences solve very little due to the fact that they are *incomprehensible* to the receiver and have a marginal function as far as ordinary language is concerned but, at the same time, through contrast, create a potent and significant use of the arithmetic and algebraic coding. The heart of the messages (here represented by the dots) is the mathematical nucleus of the speech. Brioshi has by now already been introduced to all of the classes of the Project, and contributes a potent support in the passage of an often difficult concept for 8 – 13 year olds, that being: the necessity *to respect rules* when using a language, and this becomes an even stronger necessity when the language is formalized due to the extreme synthesis of the symbols used.

This activity begins by proposing to the students an exchange (their class and Brioshi’s class) of messages written in simple Italian sentences that they have to try to translate into mathematical language; each time the different translations are copied onto the board and collective discussions take place so as to chose which sentence to send to Brioshi; then the class awaits a reply and it is then interpreted. Even Brioshi's class exchanges messages and problems which have to be interpreted and answered. This exchange can be simulated within the same class and in this case it is the teacher who proposes Brioshi's 'reply messages', or invites a student to imagine he/she is Brioshi. This 'role game' always functions, no matter what the age of the students. The exchange acquires an interest and becomes even more effective if there are 'real' classes that exchange messages – or a virtual reality class – by means of *ad hoc* email messages written (using Japanese letters) which the teacher presents to the class, of course to their total amazement and surprise. The most interesting modalities – which began in January 2001 – concern an exchange of 'mathematical communication' messages with Brioshi carried out among different schools by means of a specially set up chat line (software Net meeting).

Let us add that within the Units, other than within Unit 1: 'Brioshi and the approach to algebraic code', there are also other useful and important elements of mediation: i) analogy with typical situations of the natural language (for example the multitude of representations with which a number can be denoted other than its canonical one, just as each individual can be called with nouns other than his/her own name and a countless quantity of locutions that take into consideration his/her relatives, friendship, work) (in all Units); ii)the resource of the 'masks' for activities regarding the relational aspects of a number (Unit 4: 'Matematóca & other mathematical games'); iii) the use of the smudge or a cloud as a strategy to imagine what is hidden behind a short mathematical text so as to induce students to using letters (Unit 3: 'Numbers Pyramids' and Unit 4: 'Matematóca & other mathematical games'); iv) the islands, the archipelagos and travels used as elements of explorative games within the numbers chart/grid so as to favour a synthetic representation of a chain of additive operatives or even to innocently introduce the concept of un-determined (Unit 2: 'Numbers grids'); v) the scales used as an instrument to approach equations *(*Unit 6: 'From the Scales to the Equations').

Unit 2: Numbers grids (7 – 13 year olds)

This Unit represents a gym for pre-algebraic thinking through to actually being the area of first grade equation application. Activities are developed around the exploration of a square of one hundred number boxes from 0 to 99. Through the discovery of regularity, and games using 'numeric pathways' within and on fragments of the grid ('Treasure island', 'The island game', 'Never never land'), therefore problem situations even on grids of diverse dimension and reflections upon the different ways to represent numbers in the boxes, the Unit leads to generalization by using letters and thus at last to the 'conquest' of the grid having a dimension of n × n.

Students have to overcome difficulties concerning mental and written calculations, verbalization of mental processes, confrontations between arguments or representations, whereas older students affront arguments that secondary students hardly ever face, even though these hold an important position in the development of algebraic studies for all.

Unit 3: Numbers Pyramids (6 – 13 year olds)

The Unit intend to favour the development of *relational* thinking. By exploring the 'pyramids' made up of 3, 6, 10 bricks, students are led to individualizing and representing a more and more complex link among the numbers written within the bricks. Emphasis is given to the binary aspects of the operations and the *non-canonical* representation of the numbers. At the start the activity takes place within an arithmetic ambient and then progressively widens towards algebra and the *naive* discovery of the use of letters and equations (linked with Unit 6: 'From the Scales to the Equations'). Reflections on the representation helps to highlight the linguistic and meta-linguistic aspects (linked with Unit 1: 'Brioshi and the approach to algebraic code'). The itinerary exhorts observation, exploration, reflection and argumentation in problem situations by referring to diverse collective numeric groups. Strategy confrontation is preferred and leads to a gradual discovery of regularity, generalization and use of letters.

Unit 4: Matematóca & other mathematical games (7 – 8 year olds)

In this Unit by means of original variants of board games (Dominoes, Memory, Bingo) or by means of invented games (The masks game) students are supplied with material that obliges them to re-visit arithmetic arguments from a view point that favours an algebraic vision. At the same time, by using opportune didactical mediators (smudges, clouds, slips of paper '), etc., students approach the unknown number and the possibilities of the ways of representing it. Step by step as the games proceed, the materials that make up their concrete supports modify and the indications written in natural language are transformed into simple algebraic writings in which the unknown is represented by the score of the dice used in the game.

The approach to this Unit is a game – thus children respond to it more and it helps to create an ideal learning area. This Unit affronts various aspects, among which: the relation of being equal as a relation of equivalence and in particular the symmetric and transitive property, the partition of a set, the confrontation between different forms of representation of the same number and therefore operative properties and mental calculation exercises. In addition the use of letters as a parameter is introduced by using the score number of the dice.

Unit 5: Regularity: frames and necklaces (8 – 11 year olds)

In this Unit the activities involve the need to discover the regularity of a structure. In the first stage, students analyze necklaces made up of different coloured beads which are positioned alternately in the necklaces; in the second stage, students analyze structures that are made up of matches that form houses, bridges and field of various dimensions; in the third stage, they analyze prints and stamps; in the fourth arithmetic successions. At each stage, through exploration and discussion, students search for regularity and successively for the representation in mathematical language. The discovery of regularity is precious in forming pre-algebraic thinking, as it favours the passage to generalization (linked with Unit 1: 'Brioshi and the approach to algebraic code').

The unit starts off with the observation of regularity: in particular the analysis of the regularity of the succession of beads on a necklace and then the arithmetic successions are studied. This kind of work is ideal for discussion and ideal for constructing knowledge, and pushes students towards a level of formalization by means of 'putting into formula' (done in a naive way) the rules they have found. Within the Unit there is a large work area of argumentation: the children are asked to justify their intuitions which are often refused by their companions and their explanations and reasoning are quite often sophisticated. Brioshi is also used in this unit so as to motivate the use of the algebraic language.

Unit 6: From the Scales to the Equations (10 – 13 year olds)

This Unit approaches algebraic thinking. Through collective solutions to problematic situations and with the pan scales students discover 'the principle of equilibrium' and the two principals of equalness; the passage from an experimental activity through to its written representation leads to the 'discovery' of letters in mathematics and equations. Even algorithms for the solution to equations are progressively elaborated and refined through collective and individual activities during which students elaborate and compare diverse representations, refine their competence of natural language translations and symbolic ones and vice versa and moreover students get used to using letters as the unknown. A succession of opportunely organized verbal problems of different levels of difficulty lead students to investigating how to solve problems using algebra.

This activity approaches algebraic thinking by means of collectively building up knowledge; pupils elaborate and compare different representations, they refine their competence of natural language translations compared to the symbolic language and vice versa and moreover students get used to using letters as the unknown.

Part II

The ArAl Glossary

A mediation tool between the theoretical framework and the teaching Units

1. **Additive (form, representation)**
2. **Algebraic babbling**
3. **Argumentation**
4. **Arrow representation**
5. **Brioshi**
6. **Cancellation laws**
7. **Canonical / non canonical (representation, form)**
8. **Codification → Formal coding (translation into formulas)**
9. **Collective (confronting, discussion)**
10. **Composing the operators**
11. **Computational (aspect)**
12. **Concluding indicator → Equal (sign)**
13. **Confronting → Collective (confronting, discussion)**
14. **Denotation → Sense / denotation**
15. **Describing (in a mathematical language)**
16. **Diary (activities in the co-presence of teachers)**
17. **Didactical Contract**
18. **Didactical Mediator**
19. **Directional operator → Equal (sign)**
20. **Discussion → Collective (confronting, discussion)**
21. **Equal (sign)**
22. **Equivalence relation**
23. **Formal coding (translation into formulas)**
24. **Formal/formalization → Translating, translation**
25. **Hybrid (equation, writing, representation) → Pseudo equation**
26. **Indeterminate**
27. **Indicator (letter like) → Letter (the use of)**
28. **Ingenuous (discovery) → Symbol euphoria**
29. **Initial (letter like) → Letter (the use of)**
30. **Label (using letters as)**
31. **Lack of closing (an operation) → Equal (sign)**
32. **Language (mathematics as)**
33. **Letter (the use of)**
34. **Mathematical phrase**
35. **Metaphor → Didactical Mediator**
36. **Multiplicative (form) → Additive (form, representation)**
37. **Name (of the number) → Canonical / non canonical (representation, form)**
38. **Name place (letter like) → Letter (the use of)**
39. **Negotiating**

1. Additive (form, representation)

2 + 2 + 2 + 2 or 5 + 3 are *additive* representations of the number 8, in the same way that 2 × 4 provides a multiplicative and 2^3 its *exponential* representation. These three forms express the evolution of mathematical thinking, each one culturally including the previous one. Even students reproduce this evolution, which does not occur necessarily according to pupils' age. Initially students feel more at ease with additive representations, which are more spontaneous and reassuring, as they were studied first and have thus determined a kind of *thought imprinting*. For example: with 5th year elementary pupils (10 year olds) working with the scales (see Unit 6: 'From the Scales to the Equations') during the construction of their first equations, they are presented with a situation in which there are, on one of the pans of the scales, many similar objects of an unknown weight. In this case, we notice how the representations reflect their epistemology (a product of their personal history and their personal convictions):

1) "A group of objects"	x x x
2) Additive representation of the total weight	x + x + x
3) Multiplicative representation of the total weight	3x

However, the same situation also arises in an arithmetical context.

For example, in the following problem:

> "The teacher has bought her class 18 identical kits, each containing an eraser and a pencil sharpener; each kit costs 1,4 euros. Represent this situation so as to show how much she spent."

Some students use the multiplicative representation like 1,4 × 2 × 18, whereas others try the additive representation 1,4 + 1,4 − only to realize with difficulty that they need to insert brackets: (1,4 + 1,4) × 18. When this does not occur, the writing 1,4 + 1,4 × 18 naturally produces an erroneous result. (Sometimes the student might forget the brackets − a syntactic error – yet perform the operations as if the brackets were in place – which is known as a correct procedure – thus obtaining the correct solution anyway.)

2. Algebraic babbling

Algebraic babbling is a metaphor, elaborated within the ArAl project, which brings the learning methods for the algebraic language closer to those used for natural language. During the natural language learning process, a child acquires, little by little, a knowledge of the *meanings* and rules of his or her language. These are then gradually developed through imitation, error, invention, approximation and gratification to in-depth studies, when the child reaches school age and learns to read and reflect upon the *grammatical* and *syntactic* aspects of language.

By analogy, we would presume that the same thing should happen with the algebraic language. The 'babbling' metaphor contrasts traditional algebra teaching, in which the study of *rules* is privileged, as if formal manipulation were in some way independent from the understanding of *meanings.* Thus we propose constructing algebraic thinking *progressively and in parallel* with arithmetic thinking. This should start from its *meanings* and continue through to the construction of an environment that *informally* stimulates autonomous elaboration, experimentation and continual re-definition of a new language in which the rules can be gradually collocated. All this, of course, takes place within the framework of a didactical contract that is *tolerant* of those initial 'promiscuous' syntactical periods.

3. Argumentation

According to Webster's Dictionary, the term *argumentation* means 'The act of forming reasons, making inductions, drawing conclusions, and applying them to the case in discussion; the operation of inferring propositions, not known or admitted as true, from facts or principles known, admitted, or proved to be true'. Argumentations can be expressed verbally or in writing, by numerical data or graphical representations. An argumentation may include more than one theory, thus verification takes place through simple reasoning, immediate examples and experimental trials. Argumentation is used in heuristic activities, where conjectures are formulated, and one freely searches for solutions to problems by activating a semantic connection of the given facts, so as to underline new relationships between data and objects (the constitutive process of informal demonstration). Argumentation is the fundamental means of that constructive social process of knowledge, which should be a typical mathematical activity in elementary and secondary school teaching.

4. Arrow representation

This kind of representation has become widespread with the introduction of the theory of sets and it is usually used to represent a correspondence between one set and another. For example, let us consider a set of *children* and one of *names*; the arrow with the writing 'to each child their own name' represents *the set of pairs which verifies the correspondence* in question. This also occurs for the correspondences in the same set, for instance: the self-correspondence in the set of naturals which makes every number correspond to its triple − the so-called *multiplicative* operator ×*3* – is represented by an arrow with the writing '×*3'* that indicates which action to take in relation to the number.

Similarly with *additive* operators, for example: '+5' is the operator that to each number associates the sum of the said number to 5. The arrow representation for the additive and multiplicative operators permits one visualize their composition, which associates to each number the sum obtained by means of the successive two operatives. For example, two consecutive arrows, which represent the operatives '+ 5' and '× 2', underline the correspondence and composition which associate to each number the sum obtainable by applying the operatives.

Therefore the arrow representation underlines not only the abovementioned composition, but it also permits the passage from arithmetic numbers to a higher level in which objects become the *operatives.* In this new form of arithmetic each operative, being reversible, obtains a *symmetry* – that is an operative which acts by undoing what it has previously created – so that the composition of both remains *invariable*. For example: the operative '+ 5' has the symmetry of '- 5', and the operative '× 7' has the symmetry of ': 7'.

In this way the use of arrows is extremely useful for analysing the calculus process and the passages it is made up of. For example in the expression [(3 + 2) × 5 – 3]: 2 that has the result of 11 can be seen as the action of number 3 in the chain of operatives '+ 2', '× 5', '- 3', ': 2'. Each of these operatives is reversible, thus when operating upon the number 11 with the composition of inverse symmetry, that is by means of applying 11 to the chain '× 2', '+ 3', ': 5', '- 2', the result obtained is 3 (the result of the expression (11 × 2 + 3) : 5 – 2 is in fact 3). This vision becomes more immediate for the students if it is supported by the arrow representation.

This vision is very useful when resolving linear equations of an unknown form as in ax + b = c. In the interpretation of such an equation with the calculus process of an unknown number, it is possible to resolve the problem by starting from the form that is known. For example the equation 2x + 5 = 13 can be visualized by means of arrows as the composition of the operatives '× 2' e '+ 5' which, by acting upon the number x gives way to the number 13. Whereas, by starting from the number 13 and applying the inverse symmetry of the operatives, that is by applying to 13 firstly the operative '- 5' and then to the result adding the operative ': 2' one thus determines the value of x, which is 4.

The arrows are also used to demonstrate the order of a relationship. For example, within the ambient of natural numbers they demonstrate the relation of 'being dividable', so it is easier to identify all the division forms used and the paths taken to obtain a given number.

This representation is also used when resolving logic problems which include the order of relationships, as in 'he is bigger', 'she is taller' within the field of collective groups of individuals.

5. Brioshi

Brioshi is an imaginary Japanese student (his age varies according to his interlocutors) and he is a incredibly potent support within the ArAl project (see Unit 1: 'Brioshi and the approach to the algebraic code'). He was introduced into the project so as to aid students aged between 7 and 14 in grasping a difficult concept, that is, the necessity to *respect rules* when using a language especially when using a formalized language due to the extreme synthesis of the symbols used. Brioshi can only communicate by using a correct mathematical language and enjoys exchanging problems and solutions with classes from different nations. Brioshi communicates by using different means, from messages on pieces of paper through to sophisticated software technology (Net Meetings).

6. Cancellation laws

These are arithmetic laws whose genesis lies within the principle *If from equal things we add or take away equal things we obtain equal things.* These laws are the base of the laws of transformation of equations (see Unit 6: 'From the Scales to the Equations').

The first law involves addition (subtraction):

data a, b, c natural numbers, if a + c = b + c (or a - b = a - c) then a = b.

The second law involves multiplication (dividing):

data a, b e c natural numbers, with c different from zero, if a × c = b × c (a : c= b : c) then a = b.

Obviously the opposite can be said and applied for each law.

7. Canonical / non canonical (representation, form)

For anyone who thinks of a number, this means thinking of a certain quantity expressed by a positional representation in base 10. If the quantity is less than ten, then it is represented by one of the numbers 0, 1, …, 9. If the quantity is more than ten, then it is represented by a string of numbers each of which indicate from right to left how the sum is made up: units, tens, hundreds, etc.

This representation also gives rise to the name of the number, for example 134 is called 'one hundred and thirty four' and for this reason is called *canonical.* However a number can be represented in a myriad of ways, not only through a change of the positional base representation but also by means of whatever expression has its result. Of course each representation will have its own determined sense of the underlying process or one within its numerical reference, for example the writings 2; 2,00; 4/2; +2, $\sqrt{2^2}$, all represent the number 2. The canonical representation of each number – its proper name − is *opaque* in meaning, it does not *say much of itself* to the student. For example, the number '12' suggests a generic 'number of things', or, at best, the idea of an "even number". Other representations – depending on age – can increment instead the field of information: '3 × 4' shows that the sum is a multiple of both 3 and 4; '$2^2 \times 3$', shows that it is also a multiple of 2; '2 × 2 × 3' leads to '2 × 6' and thus to a multiple of 6; 36/3 or 60/5 that it is a submultiple of other numbers, and when it is inserted in the form of a squared root $\sqrt{2^2 \times 3}$ it helps in the passage towards $2\sqrt{3}$ and so forth.

Each of the possible connotations of a number *adds useful information* to increasing its understanding, exactly as would happen among people. Giancarlo – Alice's father, the owner of this or that camper, the third class C's mathematics teacher, the Coordinator of the ArAl project, Francesco's friend, the Italian speaker and so on, all represent the same person. They are all different view points and increase knowledge of the 'canonical form' of 'Giancarlo Navarra'. We shall see later on why getting the students used to considering a number not only '12' (its *proper name*) but also '9 + 3' or '$2^2 \times 3$' is an important step towards the solution of a series of problems and the understanding of writings like 'a + b' or 'x^2y'.

8. Codification → Formal coding (translation into formulas)

9. Collective (confronting, discussion)

International research regarding mathematical education has widely demonstrated the certainty that activities which include verbalization and discussion do in fact favour mathematical learning. The efficiency of these activities is demonstrated through writing and through the production of protocols related to problems which have been opportunely organized, thus, in various different ways, through *discussions.*

In actual fact, a standard '*discussion'* does not exist, however different forms or *models* of discussion can be created in class regarding themes connected to mathematics, which provide the teacher with different roles to in-act progressively. For example: a discussion about the collective solution to a problem *(problem-solving discussion),*or the confrontation of strategies elaborated singularly or in groups when solving a problem *(balance discussion),* or discussion about general themes *(conceptualisation discussion* or *philosophical discussion).* For example 'What are numbers for you?' 'What is geometry?'. The importance is to favour learning process activities.

From *a social* point of view, these activities stimulate the capacity to listen, to compare, to collaborate, to discuss, to measure oneself with ones' own epistemological obstacles: that is to prevent the construction of knowledge that is a result or social *meditation*. From a *cognitive* point of view these activities contribute in refining the metacognitive capacities: that is reflecting on one's own knowledge, on the often subtle differences between *knowing and understanding,* or the relationship of acquiring and using techniques and the comprehension of the theories that support those said techniques. Whereas from a meta-linguistic point of view, one reflects upon the natural language and the mathematical languages, or the syntax and the semantics of both – verbal, icons, graphics, arrows - or upon the translation of a natural language and a mathematical language.

Actually putting into practise discussions in 'maths classes' is not a very widespread custom among teachers, due to the fact that teachers are often diffident and worried about the difficulties which arise from '*how to control a discussion, its modalities and length'.* In fact, discussions and collective confronting, over the long run, do represent a potent vehicle in terms of mathematical and linguistic competences. One may therefore affirm that it is the didactical structure itself that is influenced, that knowledge within a game is more *dynamic*, and in many ways the students are involved its creation. In this sense they represent a pedagogic ambient which is decisive for the development of algebraic babbling.

10. Composing the operators

The operators are divided into additives '+5', '-3' or multiplicative '×4', ':2'. The name derives from the fact that they refer to the operations of additions and multiplications. By composing additive operators in two directions one obtains the same operator, the same can be done with two multiplicative operatives. This is expressed by saying that the composition of the additive operatives is commutative (in the same respect the composition of the multiplicative operative is commutative, too). In general, however , by composing two different types of operators the latter does not occur. For example: starting from the same number, 7, and applying the two operatives '+2' ':3' in the two ways possible, one obtains two different results (in the first case the result is 3 and in the second case the result is 7/3 +2); therefore the composition of any which operative does not provide the commutative law.

During studies, it is important to make student understand that the operators are objects which have their own 'arithmetic' compared to the operation of composing: the associating principle is what counts, there is a neutral element to take into account compared to the operation of composing and that is the operator '× 1' or the *equivalent* operator '+ 0'; for each operator there exists the opposite operator, meaning, an operator which can be composed in two directions with the first, forms the operator '+1'. It is a delicate situation that of trying to explain that the operatives '× 1' and '+ 0' are *the same* operator – it is hard for students not to take into account the representation and to consider instead the *corresponding* rule, that rule which in both cases each number corresponds to itself. It is also rather tricky trying to get across the notion that the '+3' operator is the opposite of the '-3' operator, because for every number n is: (n + 3) - 3 = n and (n - 3) + 3 = n, being the composed operators '+ 3 - 3' and '- 3 + 3' operate on the n as if it were the *neutral* operator. That is why, it could be useful to speak instead of a *symmetric operator*.

Within the activities frameworks, and in particular in Unit 2: Numbers Grids and in Unit 5: Regularities: frames and necklaces we have used arithmetic type operators. In that context the operator has been interpreted by means of a metaphor to give orders when passing from one number to another. Students have spontaneously used the arrow representations to describe the operators, for example $2 \xrightarrow{+5} 7$.

11. Computational (aspect)

This is the aspect that while studying a problem prevalently focalises on *the calculation*, neglecting the underlying thinking *process* thus rendering its logic obscure. This is common with teaching methods which focus the attention more on the *know how* than on the understanding and knowledge of the reasons that legitimate a process.

12. Concluding indicator → Equal (sign)

13. Confronting → Collective (confronting, discussion)

14. Denotation → Sense / denotation

15. Describing (in a mathematical language)

The word *describe* indicates an activity by which, through observation, we can express ourselves using a natural language. One may also use observe forms and *relationships* between them using a mathematical language. In this case the words become symbols and even the predicate are translated in relationships among forms. Just as a language offers various possibilities of describing, so does the language of symbols – at times it may have less terms that are univocal in meaning, but the student is therefore obliged to learn to recognise equivalent descriptions. There are important activities regarding this argument in Unit 1: 'Brioshi and the approach to the algebraic code'.

The phrase:

'Represent what sum is missing from 6 to obtain a total of 9'

can be translated with:

6 +…= 9 or with …= 9 - 6 or with 9 - …= 6.

These are paraphrases, which can be translated from natural language into arithmetic language; here they describe the different possibilities of translating the number 3 as a difference between 6 and 9. In order to operate using a multitude of descriptions, even within a simple situation as described here, the student must possess a sure and consolidated knowledge of the fundamental arithmetic operations.

16. Diary (activities in the co-presence of teachers)

The diaries represent a fundamental analysis instrument in the teaching/learning process of the ArAl project. The diary is compiled by the class teachers during "joint lessons" − i.e. classes where more than one teacher is in attendance − together with the researchers. The diaries register the experimental activities step by step as they are presented in class, together with the discussions and the written protocols, the surprises and the mistakes. These notes are then reprocessed in electronic form by the researchers (in the school year 2000/2001, 143 were prepared). These are then judged and valued by the ArAl teachers; and at the end of the school year they are discussed and organised within the GREM group and then transformed into Units that are tested on the classes which participate in the project. After this trial within the classes, the Units are processed one more time and refined with material taken from the diaries and offered to teachers outside the ArAl workgroup.

17. Didactical Contract

This is a theory of the French mathematician G. Brousseau (1986) and it indicates that sum of relationships a small part of which are explicit but a larger part of which are implied, that regulate the relationship between the teacher and the students in respect to the development of the knowledge of a specific mathematical content. These situations are resolved within a system of *obligations*, within which the didactical process involve both the teacher and the pupils, all of whom must absolve their duties and also be responsible in respect to their companions – that is why the term *contract* is used.

18. Didactical Mediator

The name *didactical mediator* is applied to all those instruments which help a student in his often difficult moments of grasping new concepts. In many cases, they are (or should be) spontaneous and creative support from the teacher, and they are anyway an integrated part of any learning process. The role of the mediator is that of *ferrying one across* from a familiar experience towards an unknown one by means of the exploration of perceptible *common* elements both in the situations of departure and arrival. In the ArAl project, important elements of mediation are:

i) Analogies with typical situations of the *natural language* (for example the multiple ways of representation which denotes a number other than its canonical representation – just like each individual may also be denoted, not only with his/her own name, but also with a myriad of locutions which take into consideration family relationships, friendship, work etc) (see in particular Unit 1: 'Brioshi and the approach to the algebraic code').

ii) The use of the *masks* for activities regarding relations with numbers (see Unit 4: 'Matematòca & other mathematical games');

iii) the *smudge* or the *cloud* are used as strategies to conjecture what might be hidden behind a short mathematical text and to induce the passage through to using letters (see Unit 3: Numbers Pyramids and Unit 4: 'Matematòca & other mathematical games');

iv) The island, the archipelago and the journey are elements of fun and exploration within the area of numeric charts and are structured favourably to represent synthetically the chain of additive operators or also to introduce in a naive way the concept of undetermined (see Unit 2: 'Numbers Grids')*;*

v) The *scales* used as an instrument of approach to equations (see Unit 6: 'From the Scales to the Equations').

The more significant a mediator is, the more *powerful* it becomes; however at the same time it is opportune to bear in mind its limits, which are hidden within this efficient potency. Here we underline two such inconveniences:

1) Research has demonstrated that it is possible to encounter interferences among the intrinsic characteristics of the mediator and those that it acquires when it enters and becomes part of a metaphor. For example in Unit 5: ‘Regularities: frames and necklaces’when using the *stamp* (or *print*) of a *frame* (used since pupils were at nursery school) as mediators, so as to help the understanding of the *module* of a *succession*, in actual fact, with these four elements we set up a ‘proportion’: *the stamp is to the frame what the module is to succession.* However, so as to become an active part of the metaphor, the stamp and its relative frame have to in some way *lose* their original characteristics, thus becoming a bridge towards a new knowledge (the concept of regularity of a succession).

   Until the moment that this necessary *reconstruction of the field* takes place, the mediators will only be a hindrance and end up by making the potentiality of the metaphor opaque in its function as learning instrument. In the example shown, the characteristics of the frame – connected to the esthetical aspect and pleasures of creating a real object and the sensation of being allowed to use ones’ fantasy and creativity freely – may induce in students (above all in younger students, but also in older students, as stated in some ArAl diaries) an excessive concentration of *concrete* aspects, instead of the *conceptual* aspects which should have been used.

2) The second, which is connected to the former, may manifest itself for example when working with an instrument like the scales and with representations connected to this instrument (see Unit 6: ‘From the Scales to the Equations’). It has been demonstrated that when maintaining a excessively prolonged contact with the mediator, the teacher runs the risk of conditioning the students and leads them to the reassuring use of a concrete object, which thus blocks their evolution of both thought and language. In other words, the mediator runs the risk of becoming a block or of transforming itself into a *stereotype.* In many ways these observations are similar to those developed regarding semantic persistence. In conclusion: the metaphor may be useful as a temporary pedagogic instrument which amplifies existing structures and creates a semantic link between structured learning and new information. Therefore, we wish to enhance that teachers use these efficient mediators and metaphors when and if necessary, and to use them time and again, however it is wise not to become attached to them so that students do not create stereotypes by giving too much emphasis to this support. The mediators utility terminates as soon as they have reached the objective for which they were introduced.

19. Directional operator → Equal (sign)

20. Discussion → Collective (confronting, discussion)

21. Equal (sign)

When teaching arithmetic at junior school, the equals sign essentially expresses the meaning of *directional operator* and for the student 4 + 6 = 10 means: add 4 and 6 and obtain 10. This concept remains a foothold for the first seven or eight years of school in which the equals sign has a dominating *space temporary* connotation: it prepares the conclusion of a *story* that must be read from left to right (the operations are done in sequence) through to its conclusion (when finally the result is obtained). Then, traditionally, student meet algebra at secondary school, and suddenly the equals sign takes on a totally diverse meaning: it indicates the equivalence between two quantities. In the writing like '8 + x = 2x – 5' it takes on the meaning of *relational* and it contains the idea of *symmetry between two writings*. Students must suddenly (and usually with no warning of this widening of meaning) be able to move within a totally different conceptual universe, where it is necessary to go beyond the familiar *space temporary* connotation. Therefore if a student's concept is that 'the number after the equals sign is the *result*', then probably fro him/her the writing like "11 = n" does not mean much, even if he/she can resolve the equation.

An example of this is: when requested to write "fifteen plus twelve" nearly always student write '15 + 12 ='. The equals sign is seen as a concluding indicator and it expresses how the student is convinced that sooner or later the teacher will require that conclusion. It denotes an operative behaviour, which is the harvest of didactics often concentrated on *calculations* and too much upon the procedural aspects. The absence of the sign is interpreted as 'lack of closure' of the operation (in fact many would consider the writing 15 + 12 'unfinished').

22. Equivalence relation

Reflections on equality, and in particular students' interpretations of the equal sign as a directional operator, oblige one to reconsider meanings, properties and the use of the equivalence relations. The relations in a set may have some laws; those defined as 'equivalence relations' have the following three laws:

- *Reflexive law: Each element of the set is in relation to itself;*
- *Symmetric law: If element A is in relation to element B, then element B is also in relation to element A;*
- *Transitive law: If element A is in relation to element B and element B is in relation to element C, then also element A is in relation to element C.*

These are examples of equivalence relations of a set of people: having the same height, the same weight, the same name, being born the same year. The importance of the equivalence relations is due to the fact that they are the basis of equality, which is always relative to some of these aspects (in the latter examples: equality of height, weight, year of birth). In a relationship of equivalence the elements related among themselves are called equivalent and can be identified compared to the distinctive property of the relation. In many of our Units we have operated on the set of expressions in natural numbers and we have identified those with the same result; this can be considered a prototype of all the equivalent expressions, each of which is its legitimate representative.

23. Formal coding (translation into formulas)

This is the act of translating from a natural language into an algebraic language in relation to data that is (known or unknown) within a problem situation. The most delicate aspects of such a process consist of:

(ii) the choice of the names (letters) to give to the known and unknown quantities, that make up the data of a problem;

(iii) the control of the diverse representations through which the same relationship may be expressed;

(iv) the ways in which the relations and (therefore the representations) are connected among themselves by means of substitution or confrontation.

For example :

Represent the relation: the segment $\overline{AB}$ *is larger by 4 cm than the segment* $\overline{CD}$

It may be coded in different ways:

1. $\overline{CD} = \overline{AB} - 4$;
2. $\overline{AB} = \overline{CD} + 4$;
3. $\overline{AB} - \overline{CD} = 4$.

It is wise to get students used to the fact that a good translation depends on its efficiency to favour the synthesis of all the relationships involved. The activity of translating into formulas can create difficulties. There are situations in which even the verbal formulation of the request leads to errors, due to the fact that students generally tend to translate the request word by word. A classical example of this error is:

> *Write an equation that represents the following proposition: 'In this university there are 6 times as many students as there are professors'. Use P for the number of professors and S for the number of students.*

Frequently, the translation of the proposition in the form of $6S = P$ instead of $6P = S$. This is known as the *inversion* error. It is a good idea to get students used to using a paraphrasing of the proposition they have to translate, so as to check its meaning.

In order to facilitate the translation into formulas, one may also use the texts of problems from which the question or conclusive questions have been omitted. Initially, it is advisable to indicate to the students the object or the objects they have to name.

An excellent example is the following:

Julie is 4 years younger than Dora, Elena is 3 years older than Dora. The sum of their ages is 38. Indicate with a letter Dora's age and express by means of a formula the relation between the information given. Then express the relationship again with another formula, starting from Julie's age.

It is important to guide the students towards the understanding by that expressing the same thing in different ways leads to *equality* between the writings.

24. Formal/formalization → Translating, translation

25. Hybrid (equation, writing, representation) → Pseudo equation

26. Indeterminate

This is an ancient denomination that has its roots in the period in which the first use of letters was to generalize problematic situations and when entire *classes of problems* were represented by a problem. The idea consisted in obscuring the specific values of a data by indicating them with a letter, which thus represented any given number - indeterminate in fact - a *possible* value of the data. The actual solving process of a problem was represented and summarized by the associated arithmetic expression that became – by effect of substituting the numeric value of a data with a letter – the resolution of an entire class of problems. Thus the *arithmetic* expression became an *algebraic* expression. With the consolidation of syntactic studying of algebraic forms, these two terms were classified in forms with one or more indeterminates.

Today, with a modern view of numbers, the term *indeterminate* is no longer used and has been replaced by the term *variable*, which indicates any value of an element within a data group of reference. The following is an example using the English language to explain the previously mentioned concept: one might say that the term *indeterminate*, used in reference to a generic number, is similar to the word 'each' and centres its attention on the individual; whereas the term *variable* is similar to the word 'every' which also centres on the individual but with attention and reference to the *totality* of the environment to which it belongs.

27. Indicator (letter like) → Letter (the use of)

28. Ingenuous (discovery) → Symbol euphoria

29. Initial (letter like) → Letter (the use of)

30. Label (using letters as)

Students often use a letter to indicate the *quality* of a data and not its *quantity*. For example in the translation of the sentence: 'A mix of 224 different kinds of dogs and cats live in a courtyard. The number of cats is 6 times more than the number of dogs' some students write *6c = 1d* to indicate that *for each group of 6 cats there is one dog*. In this case the letters *c and d* are indicators of the *quality* of being a *cat or a dog* rather than the respective quantities of each group.

31. Lack of closing (an operation) → Equal (sign)

32. Language (mathematics as)

The symbolic language of mathematics is extremely synthetic and concentrates within a few signs a great richness of meanings. The passage from natural language to mathematical language, essential when learning mathematics, does not occur immediately and requires lengthy exercise and gradual implementation. The natural language is more explicit, possesses words which are synonyms and offers a variety of interpretations which can be applied to the different meaning of the words themselves. The mathematical language, with all of its symbols, is more synthetic, however as it evolves it may initially represent a reading and comprehension obstacle. It also represents some advantages: symbols possess a universality that words do not have – even children who speak different languages can find moment of sharing and communicative experiences by means of the mathematical language (see Unit 1: 'Brioshi and the approach to the algebraic code'). Moreover, symbolic language has a power in the development of the thought process which the natural language does not have. If for example I write a generic odd number like *2n + 1*, (with *n* generic integer) I will be able to operate a series of reasonings about the properties of odd numbers and about the relationships of these with other numbers; however the same cannot be done with the natural language. Thus the symbolic language becomes a mediator of messages and meanings and carries out an important role in the communication and socialization of knowledge.

33. Letter (the use of)

Meeting *something that stands for* a number occurs precociously – in fact, right from primary school, even within traditional mathematical didactics. Text book that contain arithmetic exercises have assignments with - empty gaps to fill in, question marks, spaces, dots – none of these symbols are explained by the authors nor are they positioned in an attentive approach to pre-algebraic thinking. Something similar occurs within geometry with its parallel perimeter and areas formulas. However that may be, these are examples of didactical situations that remain *mute* from an educational view point and they do not in any way contribute to constructing *meanings.*

One of the core aspects of the ArAl Project is encountering the possibility of *communicating* by means of *numeric and non-numeric* symbols (icons, graphs). The approach to using letters occurs within gradual encounters with the mathematical language and it then transforms during the progressive evolution of *algebraic babbling*. This occurs within a didactical contract that tolerates the initial use of 'murky' symbols and the frequent production of pseudo-equations supported by opportune mediators which, by means of games and concrete experience, favour this approach and comprehension of letters. The intuition of the letter within the mathematical field must come as the final fruit of a collective negotiation and it depends on the environmental conditions in which each activity takes place. For example, when younger students begin exchanging messages with Brioshi, the *empty spaces* symbol is 'most' used because it seems to express best "that Brioshi has to discover the number". It may also happen that at the end of the discussion about which is the best representation to send to their Japanese friend, the class decides it is better to use an iconic writing instead of a literal one (more evolved). These aspects are important when creating the basis of semantic meanings which thus make up the algebraic language. It will be the gradual affinity of the evaluations of the students which will lead to identifying the correct translation. Concluding, we would like to mention some of the more frequent errors or misconceptions that appear in the students' work (even with older students) when the use of the letter is not supported by an opportune semantics:

- as an *initial*: in a problem, for example, that includes the word bicycle, if *'b'* is considered as the initial of the word, then perhaps in the writing '*4b*' it is interpreted as '*four bicycles'* instead of '*four times the number of bicycles';*
- as an *indicator*: this appears mixed with the iconic symbols and this concept is associated with the phenomenon of semantic persistence;
- as a *place tag*: with the only function of signalling, it substitutes an object and not the number. This usually happens in problems solved with an equation (see Unity 6: From the Scales to the Equation): the name place represents the visualization of an element of the problem situation and not its numeric value. For example: with the letter 'c' one indicates the 'box of chocolates' and not the 'number which represents the weight of the box'.

34. Mathematical phrase

This term has been modified by linguistic transformations and is now a synonym of 'writing', to indicate any kind of representation in mathematical language (expression, equation, formula, etc.). Thus, a phrase has its own *semantics* and refers to its own dictionary of signs and a syntax which determines whether or not it is correct.

35. Metaphor → Didactical Mediator

36. Multiplicative (form) → Additive (form, representation)

37. Name (of the number) → Canonical / non canonical (representation, form)

38. Name place (letter like) → Letter (the use of)

39. Negotiating

A typical class discussion process, in which the teacher co-ordinates and guides the class towards a sharing and agreement about the mathematical meanings which emerge during their exploration. This is done by activating their reflections on the pertinence of the arguments discussed or the representations expressed.

40. Opaque / transparent (in respect to its meaning)

A representation in mathematical language is made up of symbols which express meanings, the understanding of these meanings depends on the capacity of he/she who interprets that symbol but it may also depend on the actual representation of the symbol *itself*. Let us take into consideration the canonical form of a number: here we can easily say that it is much poorer in meanings that the infinite possible forms that a non canonical possesses. For example, the non canonical form $2^3 \times 3^4 \times 5^2$ gives more information about divisions of the number 16200 than the canonical form of 16200 does. Another example is the tendency to immediately carry out the calculation $2^2 \times 2^5 \times 2^3$ (probably with a calculator) thus obtaining the correct result represented in its canonical form (1024) but, alas, student lose the effectiveness of the representative 'intermediate' passage 2^{2+5+3}, which is fundamentally necessary so as to understand why in the algebraic field $ab^2 \times a^3b^4 = a^4b^6$. Therefore we may say that there is more opaqueness in writings like 16200 and 1024 and there is more *transparency* in those of $2^2 \times 2^5 \times 2^3$ and 2^{2+5+3}. In general, the transparency favours the comprehension of the *process*, that is, the modality through which a certain *product* is reached – thanks to underlining the adopted strategies, possible errors and eventual underlying misconceptions to the solution of the said problem. The opaque/transparent dichotomy represents one of the bonds through which the Brioshi project develops (Unit 1: 'Brioshi and the approach to the algebraic code').

41. Paraphrasing

To paraphrase is to reformulate the meaning of a word or a sentence with the intention or of making its meaning clearer or more simple. Mathematical education research has strongly underlined the relationship between the capability to paraphrase a text correctly and that of translating it in a correct algebraic manner. One presumes of course that the paraphrased text does not *alter* the sense of the phrase and thus helps to identify the *logic structure* – that structure which, once understood, permits the student to represent the problematic situation by means of a translation from the languages. The role of paraphrasing is fundamental when facing mathematics as a language, above all due to the fact that paraphrasing imposes the redistribution of the components of the original phrase, yet at the same time maintaining (or even exalting) the *relations* which connect them. One might say that the translation of a problematic situation in algebraic language is the *description of the relational aspects of the information that it contains*. The phrase ‘The number of cats is double that of the dogs’ requires the student knowledge of paraphrasing the second part of the phrase into ‘The number of cats is the same number of the dogs multiplied by two’; therefore the student can then correctly identify ‘c = d × 2’.

However, a first-year secondary pupil (11-year-old) translated letter by letter − a common occurrence among bad translators: ‘cats = d. of D.’ (double of dogs). It is obvious, among the various reflections that one might make about this translation, that whilst ‘d × 2’ demonstrates the actual passage from a (natural) language to another one (algebraic), ‘d. of D.’, it is however a form of personal shorthand which demonstrates not only that the author has remained *internally* within the same (natural) language, but also that he/she does not even possess the idea of *universality* (and thus does not have the comprehensibility) of a formal language. At the other end of the spectrum we could collocate another representation: ‘c/d =2’, which demonstrates that the student has grasped a *hidden* relation, that is the rapport between the two numbers. It goes without saying that students do not spontaneously activate the strategy of a paraphrasing text – this spontaneous approach usually happens without the students awareness of the fact. Favouring the use of paraphrasing is not due to an increment of competency in this area; it is due rather to leading the students to reflecting on the efficiency of this competence so as to *acquire the keys to algebraic reading* of problem situations. The character Brioshi and Unit 1: ‘Brioshi and the approach to the algebraic code’ have been finalized for exactly this goal.

42. Polynomial representation (of a number)

The positional numeration system enables one to write infinitive numbers using a finite number of digits. The base form usually used is base 10, which uses digits from 0 to 9, and permits one to write the number as a sum of the powers of 10. An example: 327 is the abbreviated form of the quantity $3·100 + 2·10 + 7·1 = 3·10^2 + 2·10^1 + 7·10^0$. This writing of the number 327 creates its polynomial form. The use of base 10 is conventional: any base can be used thus changing the number of digits used. For example in base 5 one may write:

$$1203_{\text{five}} = 1·5^3 + 2·5^2 + 0·5^1 + 3·5^0$$

yet again the number is expressed in a polynominal form.

In Unit 2: ‘Numbers Grids’ the polynomial writing of a number with two digits in base 10 enables one to identify that number within the chart, either working in base 10 with a chart sized 10 × 10 or by modifying the numeration base and the dimension of the chart.

43. Pre-algebraic (thought)

The ArAl project is collocated within that theoretical frame known as ‘early algebra’ (early start to pre-algebraic thinking). Within this frame it is assumed that *the principal cognitive obstacles are collocated in the pre-algebraic area, and that many of them spring up unsuspectingly in arithmetic contexts and create future conceptual obstacles which usually cannot be overcome during the development of algebraic thinking.* Numerous recent international studies regarding algebraic didactics demonstrate that students lack appropriate arithmetic structures from which they can generalize; moreover, they also lack the knowledge of arithmetic procedures and how these are formed, thus students do not possess a conceptual base from which to build up their algebraic knowledge.

44. Principle of economy

It is important that students get used to the idea that mathematics aims at *unifying the study of similar situations so as to resolve them in a unitary way.* An example from mathematical history: the introduction of letters to represent the data value of a problem led to unifying problems which had different data values but were *structurally similar*, thus arriving at the genesis of algebraic expressions as the *synthesis* of a solving process for each of them.

45. Procedural

This is an attribute that expresses the character of the procedure to carry out or analyse. It is also used with the meaning of sequential, in reference to the letter and local interpretation of a written text.

46. Procedure

Formal writing in which there is an ordinate sequence of instructions that are coded in a determined language. For example an arithmetic expression, containing symbols and numbers, is a procedure expressed in arithmetic language. When reading and understanding the procedure, the interpretation of the symbols is very important, so that students can grasp the order with which the instructions are to be carried out.

47. Process / product

This, together with representing / solving, is one of the fundamental qualities in the theoretic structure of the ArAl project. The representation of the *process* highlights the relations between the elements involved and provides a translation in mathematical language. The *product* is the final act of the process, and often its conciseness does not allow one to understand how this product was reached. This duality is linked to another duality which sees transparency of the process the exact opposite of the opaqueness of the product. However in mathematics the two components are aspects of the same medal: $8 \times 2 + 5$, for example, can be understood as a calculation process and as a number. It is important that students understand that these two viewpoints exist and how to distinguish them. Yet in schools, they receive very little attention within traditional mathematic didactics, and teachers usually give their attention to operative aspects that *solve* a problem and are linked to the *operations* and the result. By not taking into account correctly the conceptual aspects linked to the process means losing one of the most significant occasions that favours the development of algebraic thinking. For example, student should gradually be led to see the writing $n \times 2 + 1$:

- in a relational concept like *sum of the product of n with 2 and 1,* which expresses, on an interpretative level '*meta,'* the property of being an odd number;
- towards a structural concept, like a mathematical object or, rather, as a formal algebraic expression;
- on a more evolved level, like for example a particular class of functions: the linear (polynomial) functions.

From a didactical viewpoint, we feel it is stimulating to propose a situation that has proven itself to be efficient both with children and with adults when trying to explain this duality. We refer to the cartoon film *"Chickens on the run"*. The horrible owner of a chicken farm is upset about the low production of eggs and decides to change product. She buys a huge machine (imagine three enormous containers lined up); at one end of this machinery we watch as a live chicken is thrust into the contraption, only to exit at the other end of the machine as a horrid package of cooked chicken.

During the film, the star chicken of the cartoon is thrust into the infernal machine and, of course, her hero and co-star chicken flings himself into the machine in an attempt to save her. In the following exhilarating five minutes, the spectator watches what actually happens within the machine (feathers plucked, skin removed, sauces, vegetables, cooking, etc.). The conclusion of the example is evident: first the spectator had the possibility to see the *product* (without realizing how it was obtained) then the spectator watches the development of the *process* (interrupted, of course, by our hero).

48. Product → Process / product

49. Protocol

The use of this term is very widespread – mutated by didactical mathematical research – to indicate the written production of a student in relation to a given theme. Its vaguely notary reference and origin probably dates back due to the protocol page where students *register* their reasoning, procedures, and hypotheses by which illustrate their answers. However a protocol can bear more or less significance not only depending on the capabilities and knowledge of a student but also depending on the nature of the test and the characteristics of the underlying didactical contract. Tests which aim at merely verifying ability (the search of *results*, the application of algorithms as for example: solving an expression or an equation or applying a geometric formula) encourage the pupils into the habit of producing protocols which are not generally interesting as far as interpretation goes. Tasks of a metacognitive type (where students have *to argue* their response) lead to more meaningful protocols, in relation to both their authors' projects (and therefore their convictions, difficulties and control of knowledge) and the teachers' expectations.

50. Pseudo equation

Together with the term *hybrid equations*, the term *pseudo equations*, coined by the researcher J.T. Da Rocha Falcão, indicates the various ways 8 – 10 year olds who have not yet been introduced to algebra manage to solve algebraic problems thanks to an early and suggestive representation - that can be spontaneous or more frequently it is an opportune representation introduced by the teacher. The solution to the problem is obtained through the elaboration of *hybrid equations*, in which there is the coexistence of natural language, iconic language and formal mathematic operators. Naturally the pre-equations are not manipulated by pupils in an algebraic way, in that their function is that only of guidance in which arithmetic operation is necessary.

The pseudo equation is the result of a methodical and systematic approach, which first gets students used to representing the problem and then *solving* it. These researchers have demonstrated how this behaviour is found in young students (6/7-year-olds) and can be placed in the *zone of proximal development* related to algebraic representations, where the indications and suggestions from the teacher play an important role in attributing a sense to algebra and identifying problem-solving strategies.

Concluding: even this research consolidates the hypotheses that the important factor in favouring pre-algebraic thinking can begin much sooner than it does traditionally, with 12 – 13 year-olds. However the class must arrange a social-cultural context during the mathematics activities in which the principal aspects of the conceptual fields of algebra may be explored.

51. Regularity

There is a constant search for regularity in all Units of the ArAl project, in particular in Unit 2: 'Numbers Grids' and in Unit 3: The numbers Pyramids. The whole of Unit 5: Regularities: frames and necklaces rotates around this theme. The activities where one must discover the regularities of a structure are important in the formation of pre-algebraic thinking, due to the fact that they favour the passage to generalization: teaching someone to capture a situation of regularity means teaching him/her to identify the algebraic key to reading a certain structure. Searching for regularity provides the teacher with a lot of information: it enables him/her to understand whether the students are prepared to face problems by using methods and systematic rules, whether they can express themselves using appropriate language (using formulas), and whether they can formulate forecasts and check their validity.

52. Relational (thought, reading, aspect)

We talk about relational reading of a phrase written in mathematical language when the attention is focused not so much on its elements, but rather more on their reciprocal relations. The reading of an equation, for example, is an important moment for a relational view of writings (see Unit 6: 'From the Scales to the Equations'). Let us consider the following two problems:

On a balanced scale there are: on one of the pans a packet of sweets and a weight of 30 g. On the other pan there is a weight of 180g. Represent the situation so as to find out the weight of the packet of sweets.

Alfred climbs up onto a chair, which is 30 cm high, thus becoming as tall as his father, who is 180 cm tall. Represent the situation so as to find out Alfred's height.

Even though both problems are so different, they are solved by the same equation:

$$x + 30 = 180.$$

The solution does not appear connected to the nature of elements of the problems (in this case weights and heights). What enables us to determine the value of the unknown is the *relationship of equality* that has been written. Grasping the relationships separately would not be sufficient. The same writing, without the equal sign, would not be sufficient to determine the solution of the two problems.

Usually writings containing equalities or inequalities are read in a procedural way, that is, from left to right. For example: $3 + 2 < 5 + 2$ is seen as '3 plus 2 is less than 5 plus 2' or: $4 + 5 = 9$ is seen as '4 plus 5 is the same as 9'. It is important not to get students used only to this way of reading; they should be enforced to read the relationships in both directions '5 plus 2 is more than 3 plus 2' and '9 is the same as 4 plus 5'.

The writing of a natural number as the product of prime numbers is another example: in the writing $40 = 2^3 \times 5$ not only do all the divisors of 40 appear but also through the analysis of the relations among them one may establish if it is a multiple or a divisor of other numbers; its canonical representation (40) does not supply this wealth of information. In the Units of the ArAl project there are a multitude of situations in which the same number is expressed in different additive or multiplicative forms: identifying writings such as 3×2, $4 + 1$, $5 + 0$ forms an important step towards a relational vision of numbers, even within an elementary ambient.

53. Relation

A relation in mathematics expresses a *link* between elements. In a natural language, for example in the sentence 'Janet is Franck's son', it is the possessive 's that creates the link. Of course, in the mathematical language the definition of *relation*, and the representative modalities in which it can be expressed, have their specific functions. These relations are complex to analyse due to the *multitude of similar meanings* and the large number of possible representations.

For example, a relation can express a link between *elements of the same set* (therefore between elements of the same nature, numbers of the same type, geometrical figures, etc.) or between elements that *characterise two different sets.* In the first case, the relations defined in a set assume a particular interest when they posses certain properties, especially if they result in being a relation of order or equivalence – thus generating a particular criteria for numbering of identifying some of the elements of the said set (see Equivalence relations).

In the second case, when the relations define the link between groups of different nature, they may have the characteristic of being a *function*, thus giving space to other interpretations of this link in different areas of mathematics (for example in graphic interpretation of a *functional* relation).

Within the units of the ArAl project there are numerous situations in which the students are stimulated to try and identify *relations* and to *describe* them (there are frequent examples of definite relations within the set of natural numbers: additive, multiplicative, divisibility relations). The functional aspect can also be recognized in the sequences, because every term corresponds to the natural number that marks its place. In a more general sense, the functional aspect can be found in all of those situations in which a mathematical element is coupled to a single correspondent (for example a quadrilateral and its area).

Relations are presented in various representation registers: verbal representations, arrows representations (using graphs and arrows which indicate correspondences) double entry charts, etc. These representations enrich the concept of relation: didactically it is wise to use this wealth by stimulating and analysing diverse representations of the same relation, thus contributing significantly to enhancing the concept of a cultural instrument, so pervasive and unifying of the entire mathematical discipline.

54. Representation

Mathematical concepts are formed from abstractions, through *perceptive experiences*. Therefore to give them a foundation – even at an elementary level –it is necessary to use *representative instruments.* Thus, representations in mathematics have the explicit function of clarification in the formation of concepts.

There are *internal* representations, which correspond to mental images - around which concepts are formed and articulated – but, for internal representation to be formed, an important role is played by *external* representation. For external representations we intend the collection of representative forms of every kind (linguistic, graphic, iconic, etc.,) that are used to mediate mathematical concepts. Obviously the first and most important external representation passes through natural language. The process of explaining mathematical experiences by means of the natural language has a clarifying function and a fundamental role of comprehension. Nevertheless, the natural language is often insufficient or inadequate. The graphical or symbolic languages fulfil their synthetic functions and are powerful and effective when completing or amplifying the correct construction of a mathematical concept. When teaching mathematics, it is extremely important to use − and consequentially to get students to use independently − a multitude of representations within a field of experiences, or of the same concept.

According to the French semeiologist R. Duval, the real interior understanding of concepts only occurs when the student manages to use and co-ordinate various representative forms of the same concept, moving from one aspect to another. An immediate example of such can be had if we consider the varies ways of representation which enable one to illustrate the order of natural number, or the various uses of the method of coordinates and the Cartesian plan which illustrates the <u>relations</u> among many different elements.

55. Representing / solving

This is one of the main dualities of the theoretical framework of the ArAl Project. *Solving* a problem means aiming at the identification of the <u>*product*</u>, that is the operations that allow the identification of a <u>*result*</u>. *Representing* a problem means aiming at the identification of the process, that is the <u>writings</u> that permit us to explain in a <u>mathematical</u> <u>language</u> the <u>relationship</u> between the problems’ elements: in the first case, we privilege a <u>procedural</u> point of view, whereas in the latter a <u>relational</u> one is privileged.

This is a base concept so as to understand the passage from an *arithmetic* way of thinking to an *algebraic* way of thinking which is present in each of the Units (in particular in Unit 1: ‘Brioshi and the approach to the algebraic code’ and in Unit 6: ‘From the Scales to the Equations’). The texts of the problems which appear in each Unit constantly underline the need to ‘represent’ - which is often accompanied by the specification ‘for Brioshi’ – so as to make clear the necessity to <u>translate</u> the problem into a <u>mathematical</u> <u>language</u> and that students need not worry about *searching for operations*. An excessive concentration on behalf of the teacher regarding the modality of solving problems (for example the use of the block diagram or a rigid list of operations) may determine in the students a consolidation of the stereotype ‘solving a problem means *doing operations'.* Likewise, text books often induce this same misconception (for example, when they classify problems as 'two-operation problems', 'three-operation problems' and so forth).

Due to the general importance of these themes, we advise readers to fortify their knowledge by reading the *Theoretical Frame of the ArAl Project*.

56. Result → Process / product

57. Semantic persistence

This term indicates a misunderstanding linked to the use of symbols in the mathematical language, and frequently appears in the initial phases of constructing the algebraic language. This mis-understanding manifests itself above all in activities which approach equations, (see Unit 6: 'From the Scales to the Equations'), when student get 'attached' to iconic symbols to represent unknown quantities in problem situations where boxes, envelopes, packets, bags and so forth are used (these names are similar to the linguistic form of *collective* names). The symbol (rectangular or squared) is spontaneously introduced by the students; it seems a step towards abstraction, but actually it conserves in *itself* the strong *memory* of the analogy between its *form* and that of the object represented. So for the student, step by step, the rectangular icon ends up representing 'the envelope' and not 'the number of figures the envelope contains', the 'box' and not 'the quantity of kilos of fruit in the box', the 'bag' and not 'the weight of the nuts in the bag'. This persistent *original* meaning risks getting lost in a '*traditional*' analysis of pupils' protocols, and favours a strong misconception in the passage towards the unknown. Even the use of letters may create similar misunderstandings (a letter as the *initial* of the word, for example the letter 'c' for 'container' instead of for 'the number of objects in a container') but this seems easy to overcome.

58. Semantics / syntax

As in every language, even the mathematical language has its own *grammar rules*, that is, a set of conventions that enable one to correctly construct phrases. These may sometimes vary slightly: Italian separates the co-ordinates of a point by using a comma, whereas in other countries a semi-colon is preferred so as not to create confusion when decimals are used. The language also possesses a *syntax,* that supplies the conditions – or rather the rules − for establishing whether a succession of linguistic elements is 'well formed' (for example, the following are syntactically incorrect phrases: "9 + + 6 = 15" or the classical chain of operations added one after the other as in "5 + 3 = 8 : 2 = 4 + 16 = 20"). Language also includes *semantics*, that enables one to interpret symbols (within a correct syntactical sequence) and to then be able to establish if the expressions are true or false (for example, the phrase "1 + 1 = 10" is true or false according to the base of calculation; it is false in the base 10 system yet true in the base 2 system).

Which of the two analyses – syntactical or semantic – should come first? We can understand this by reflecting on the meaning of the sentence:

"Alice's fine"

This can be interpreted in two ways:

"Alice is fine"

"Alice's fine"

The sentence leads to a diverse syntactical analysis:

Alice *(subject)* is *(verb)* fine *(adjective)*

Alice's *(possessive case)* fine *(noun, e.g. a parking fine)*

The point we would like to get across here is that the *syntactical analysis necessarily follows the semantic analysis*, so the implications of this statement are extremely relevant.

In the perspective we are considering, the *translation* of phrases from a natural language (graphic or iconic) into a mathematic language − and vice versa − represents one of the most fertile territories within which one may develop reflections regarding the mathematical language. Translating in this sense means interpreting and representing a problem situation by means of a formalized language or, on the contrary, recognizing within a symbolic writing set the situation it describes.

This demonstrates how semantics is subordinate and leads to a syntactical analysis. In mathematics learning, where there is a continual exchange between verbal language and <u>mathematical</u> <u>language</u>, it is necessary for students to activate a control using two expressive registers and their metacognitive abilities of comprehending how the transformation of formal syntax expressions condenses difficult thought processes which would otherwise be arduous to complete in a natural language.

59. Sense / denotation

The distinction between *sense* and *denotation* of an expression can be understood in the semantics of Frege. His classical example is the double denomination of Venus as *Vespero* (night star), and *Aspero* (morning star) denominations with opposite meanings. When dealing with algebraic expressions, the term *sense,* means explication of the way in which the denoted can be obtained by means of applying <u>computed</u> rules. Denoted means the value or the group of values represented by the expression in reference to a given numeric form.

60. Sharing → Collective (confronting, discussion)

61. Smudge → Didactical mediator

62. Social (conquest, construction) → Collective (confronting, discussion)

63. Social Mediation → Collective (confronting, discussion)

64. Solution → Representing / solving

65. Solving / representing →Representing / solving

66. Statement

This term indicates a sentence which has a full meaning in relation to a given situation. When teaching mathematics, it is wise to give adequate space to exploration activities that lead students to expressing their observations through correct and complete statements. These should be as *correct* as possible and not *reduced* from a linguistic viewpoint, but explicit and *thorough* from an observational view point.

These statements on conjectures formulated regarding a given situation can either be capable or not of being *demonstrated*. If it can be demonstrated, the statement, it is then called *theorem.*

67. Structure, structural

Mathematics tends to unify the study of situations that present certain similarities, which can be more or less similar depending on the situational context, type of elements involved and their numeric value. The recognition of said similarities occurs by creating a correspondence between the elements of the various situations that respects their relationship. This process is known as *analogous reasoning.* When one manages to establish this kind of correspondence, the situation is *analogous* or presents the same structure, or between them there is an *analogous structure.* The term structure means the net of relations that flows among the elements of a same situation. Situations can be deemed analogous when they have in common this net of relations. A typical case of analogous structure is found in problems. In fact the term *structure of a problem* is used in reference to reasoning, that by means of linking data together enables students to resolve the said problem.

To identify and classify the diverse structures we speak of *problem models*.

Let us consider the following problems:

1. *To make a skirt the seamstress needs 1.20 m of material and as much lining. The cost of the material is €73 per metre and the cost of the lining is €12 per metre. How much does it cost to make the skirt?*

2. The height of an isosceles trapezium is 21 cm, and half of each of the bases is 35 cm and 24 cm. What is the area of the trapezium?

3. Peter wants to invite five of his closest friends to his birthday party. He makes a deal with his mum: she will prepare and pay for the cakes and sweets and he will pay for the Cokes and pizzas. Peter discovers that a Coke costs €0.45 and a pizza costs €0.78. How many euros will Peter spend?

Even though these are different problems (due to formula, context, data type, data values) the reasoning chart is the same:

1. (cost of material + cost of lining) × length of the two pieces of material
2. (half measurement smaller base + half measurement of larger base) × height measurement
3. (cost of pizza + cost of Coke) × number of friends.

With the coming of modern mathematics the meaning structure has taken on a specific meaning in relation to the various areas of mathematics: for example algebraic structures, order structures and so on. These denominations collect the various models of internal definite groups, respectively of satisfactory binary operations of certain properties (in the case of algebraic structures) or of certain order relations (as in the case of order structures). Particular importance is given to the *algebraic structures* which permit one to understand the underlying structures in the various numeric areas (natural, internal relatives, rational, real) and to recognize the analogous structures of these groups in respect to other non-numeric fields.

Here the term structure means the net of relations that exists between elements of a group which are together, due to the properties that the operations among themselves each possess. The importance of this vision is that attention is moved away from the number and its actions, and is focalised thus on the laws that govern the group of ownership, permitting one to pass from a <u>procedural</u> vision (a calculus type) to a more global vision which is linked to the <u>relational</u> aspects among numbers.

When shifting attention one can recognize that with an addition operation, natural numbers are structurally analogous to the no-nil naturals with a multiplication operation, and that this creates strong similarities between order and division and yet again strong similarities between the processes that generate wholes and rationals. An even more important aspect of this kind of vision is overcoming the idea that operations exists *only* among number, conceiving therefore to operate with arithmetic rules in mathematic areas of *non* numeric type. For example, groups are formed by using union and intersection operations, the formation of arithmetic operators, exchanges, geometric transformations etc., occur by acting successively on the terms that these act upon. The laws that govern these operations are the same as those which govern arithmetic operations (associative properties, in some cases commutative properties, and in the case of two operations, distributive properties of one operation in respect to the other).

This widening prospective, that unifies and simplifies the study of areas totally diverse in the nature of their elements, responds to a typical mathematical requirement: that is to economize thinking. Thanks to an analogous structure of the two fields it is possible to transfer from one field to another those properties that are easier to recognize in one of the fields.

68. Symbol euphoria

We have decided to call "symbol euphoria" the students' frequent and variously expressed behaviour in their initial approaches to algebra, which shows their unconscious use of letters in mathematics. The letter is not seen as a number but as an *indicator* of the object represented, usually the *initial* of its name, like a *label* (in this case it is not isolated but is inserted in a group of letters which are often accompanied by a full stop, because they are in fact considered as an *abbreviation*).

It could be inserted in place of missing data or as a symbol for unknown data. For example, in Unit 6: 'From the Scales to the Equations', a problem is translated with the equation $a + 70 + 30 = p$, where the letter a correctly indicates the (unknown) height of a child, whereas the letter p indicates the (known) height of the father. The correct equation should be $a + 70 + 30 = 180$, however the student gets carried away by the novelty of being able to use numbers and thus loses the control of the significance of the writing (we would like to underline here that the letters enter even the younger students' symbolic scenario by means of geometric formulas; however in this case the letters are confined to a reserved area which could be referred to as semantically deaf).

This 'murky' use of the letter can however be a goldmine of ideas if it is inserted correctly within the context of algebraic babbling. From this point of view, it underlines an experimental base behaviour which, if adequately stimulated and guided within a didactical contract which is tolerant towards the students' *ingenuous* discoveries – may prelude an authentic conquest of meanings on behalf of the students. Hereby we collocate ourselves within the theory of potential development of Vygotskij – who argues that a child's superior physique-intellective functions appear twice during a child's growth: the first appearance is during *collective and social* activities; the second is during *individual* activities. That which a child can do today with the help of an adult (the area of his/her potential development, also known as 'proximal development zone') he/she will be able to do alone in the future. With the potential support and development of the language in its passage from prevalent means of *external* communication to *internal* mental function. The Vygotskijan theory concludes that the only form of good teaching is that which *precedes development.*

69. Syntax / semantics → **Semantics / syntax**

70. Translating, translation

Translation is an operation of *conversion* from a register of language representation to another representative register, for example, in another language. Mathematical *translations* occur frequently when passing from a verbal representative form to a symbolic one by means of a specific mathematical language; or vice versa, from a symbolic representative form to a linguistic one by means of verbal language (see all Units, but in particular Unit 1: ‘Brioshi and the approach to the algebraic code’ and in Unit 6: ‘From the Scales to the Equations’).

It is important that the gradual introduction of the symbols leads to a vision of mathematics as a true and proper language, with its own semantics and its own syntax and thus profoundly different from the natural language. Notwithstanding that the natural language is systematically used to speak about mathematics, it is necessary that students understand that the mathematical language possesses a specific formation which creates and element of *rupture* in respect to the natural language. Students should be lead to reflect upon these differences and should be aware of the use of the terminology and symbolism of the mathematics discipline.

71. Transparent → Opaque / Transparent (in respect to the meaning)

72. Unknown

It is normal for students to encounter, right from their first elementary school years, open sentences which oblige them to identify a missing element within a relationship of similarity between two terms in which one is expressed through an operation. The most frequent symbol in this case is *the space* to fill in. Number substitution symbols appear from the start in the panorama of mathematics at primary school but pupils do not interact with them, they only use them as *static* supports.

By analysing the protocols of ArAl primary school teachers working on finding solutions for algebraic problems, we have been able to observe many similar characteristics among them: they do indeed introduce iconic symbols to substitute unknown quantities, but not in the form of representations of the unknown. They are introduced as a support for *the visualization of the situation*, which is often full of expressive connotations like vases of flowers, children, bags, a sort of *semantic memory*, a reminder. In this way the iconic symbol is sometimes accompanied by a question mark and/or complete words or abbreviations, so as to strengthen the previous interpretation of symbolization as a *visualized situation.* Moreover, this is rather a *murky* use of symbols, indeed some of them may even change during the development of a procedure, which demonstrates once again that the icons are used in this case more as *signs* rather than mathematic symbols. Thus one may say that icons mark the silent reasoning process of a solver and help him/her to take *brief ‘notes of their journey’*. The two examples allow us to understand how a mathematical knowledge not educated to the correlation between arithmetics and algebra and based on culturally impoverished text books, actually hinders those favourable opportunities students may have to grasp even in traditional arithmetic teaching. In teachers’ training and practice, the introduction of the letter − and in particular, the concept of ‘unknown’ – is collocated in the moment of meeting algebra (in the true sense, with a capital ‘A’) in the last years of secondary school, usually through incomprehensible formalized memories which are totally opaque in meaning. The naive discovery of how to represent an unknown number is not part of traditional mathematics teaching. The literal symbol is *delivered* to the student when he/she is older, thus, for many reasons, when it is already too late.

However, in the ArAl project, and in particular during the construction of the mathematical language by means of more and more evolved forms of babbling, the conquest of the representation of the unknown is the moment when its significance is understood and constructed. Pupils in the first years of elementary school propose their personal representations (geometric icons, fantasy icons, expressive forms, letters, boxes, dots, empty spaces, question marks and so on) and explore their potentiality by means of comparing and reflections during the collective discussions. Some symbols evolve productively, others disappear due to their recognized inadequacy. The crucial didactical mediator is represented by Brioshi, who *certifies* the transparency of the adopted symbol. It is extremely interesting, during the slow construction of the unknown, especially in younger students, their use of the didactical mediators like the smudge, the cloud, the mask (see Unit 4: 'Matematòca & other mathematical games').

73. Verbalize, verbalization

Verbalization can be considered the process of *describing*, by means of natural language, symbolic messages or more generally of a coded process through a sequence of formal operations. Phrases of the symbolic language can be verbalized and therefore the resolution procedure of a problem.

In problem solving activities, widespread research has demonstrated the importance that students go through the verbalization process, so as to promote and increase the realization of their choices and to acknowledge the meaning of the procedures they have followed. Naturally, the incorrect verbalization of a procedure also stimulates a critical revision of the same process on the part of the student: it is a metacognitive activity, which − if used systematically − certainly leads to a larger control of his/her resolution capabilities.

74. Writing (mathematical) → Mathematical phrase