Unit 3

Numbers pyramids

$2^{nd} - 3^{th}$ grade

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Project ArAl	U3: Numbers pyramids

1. The Unit

The ***numbers pyramid*** represents a widespread field of experience in many mathematical cultures. It is a scheme generated by a pair of bricks positioned in a line next to one another, with a third brick positioned on top. Two numbers are placed within the lower two bricks whereas their sum, or a product, is placed in the upper brick.
The numbers pyramids activities represent a field for the application of equations and a mental gym for **pre-algebraic thought**.

2. Teaching Aspects

The unit intends to favour **relational** thought: the exploration of a simple yet rigid structure, like the numbers pyramid, leads to the identification and the **representation** of a network of more and more complicated links among the numbers written within the bricks. The binary aspect of the operations and the **non canonical** representation of the numbers is also emphasized here.
At first the exploration takes place within an arithmetic ambient, which then slowly spreads towards algebra and the *naïve* discovery of the use of **letters** and equations (ref: Unit 6: From the Scales to equations). By reflecting upon the representations we also evoke the linguistic aspects (ref: Unit 1: **Brioshi** and the approach to algebraic code).

3. General aspects

- The activities are very varied and the Units may take place in all classes by means of following the table 'Phases and Situations' (Par. 6). The basic 'rules' to build a numbers pyramid are simple, and do not vary as the pyramid gets larger. The understanding of these rules is simple enough even for those students who encounter difficulties comprehending verbal messages, and who often do not manage to give correct answers; not because of logical difficulties but rather due to language comprehension problems.
- From the start, the activities are based on *problems of increasing complexity* which the class, individually or in groups, try to resolve. The **verbalization** and group comparison of the strategies adopted creates a widespread consolidation of the 'discovered' results.
- The class **discussions** regarding the solution processes become extremely important here, because they force students to reflect upon their own mental process, to verbalize their own thoughts and strategies and to listen to others, thus contributing not only to the cognitive aspects but also to the linguistic aspects.
- The problems are stimulating and are presented as games or as intellectual challenges.
- Inventing new or similar problems to those already tested in class, is not a difficult task.
- The unit contains problems with singular solutions, multiple solutions and infinite solutions or impossible solutions, so as to eliminate the stereotype of problems with a singular solution.

5. Terminology and symbols

Phase	Sequence of situations of growing difficulty referred to the same subject.
Situation	Problem around which individual, group or class activities are developed.
Expansion	Hypothesis of work on a possible expansion of the activity towards an algebraic direction. Its realisation depends on the environmental conditions and on the teacher's objectives.
Supplementary activity	Enlargement towards subjects related to those developed in the preceding Situations.
Note	Methodological or operational suggestions for the teacher.
	In the square a problematic situation is proposed. The text is purely indicative; it can also be presented as it is, but generally its formulation represents the outcome of a **social** mediation between the teacher and the class
Representation	An underlined word in boldface type highlights a link to a subject illustrated in the Glossary.

Square containing the outline of a typical discussion; the following symbols may appear:

√	Intervention of the teacher
	Intervention of a pupil
	Summary of several interventions
	Summary of a collective discussion (a principle, a rule, a conclusion, an observation, …)

6. Phases, situations and subjects

PHASE	SITUATION	TOPIC
First	**1 - 5**	Activities concerning ‘arithmetic’ pyramids with two bricks as a base.
Second	**6 - 9**	Activities concerning ‘arithmetic’ pyramids with three bricks as a base
Third	**10**	First meeting with ‘algebraic’ pyramids
Fourth	**11 - 15**	Activities with ‘arithmetic’ pyramids which use four bricks as a base
Quinta	**16**	‘Algebraic’ pyramids with four bricks as a base
Sixth	**17 - 20**	Towards generalization
Expansion	**21**	Larger pyramids; Pascal triangle
Expansion	**22 - 26**	Activities with odd and even numbers

6. Distribution of the situations in relation to the pupils' age

The distribution represents an indicative proposal based in the experiences gathered throughout the years; every situation can be restricted or widened according to the methodological choices or to the opportunities of the teacher.

<table>
<tr><th></th><th></th><th></th><th colspan="26">PHASES AND SITUATIONS</th></tr>
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<tr><th>School</th><th>Age</th><th>Cl</th><th>1</th><th>2</th><th>3</th><th>4</th><th>5</th><th>6</th><th>7</th><th>8</th><th>9</th><th>10</th><th>11</th><th>12</th><th>13</th><th>14</th><th>15</th><th>16</th><th>17</th><th>18</th><th>19</th><th>20</th><th>21</th><th>22</th><th>23</th><th>24</th><th>25</th><th>26</th></tr>
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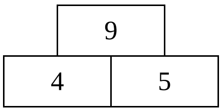
Suitable age related activities	**6**	**7**	**8**	**9**	10	11	12	13	*Comments*

First phase

1. The activity is presented to the class: with some bricks some pyramid shaped constructions can be made. Two bricks are placed next to each other with a third brick placed on top. The ‘rule’ is that the top brick contains the sum of the two lower bricks.

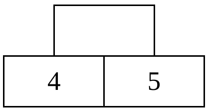
This three brick pyramid is called *mini pyramid*. [1]

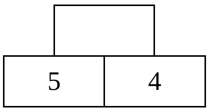


[1] *Instead of giving the rule immediately one can begin by proposing some* mini pyramids *which include all three numbers and thus ask student to identify the rule. The next step will be a* ***collective*** *class wording of the rule.*

2. We continue by proposing another problem: [2]

> If we interchange the numbers in the two lower bricks, does the sum of the top brick change, or not?

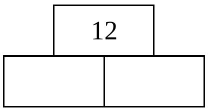




[2] *This problem favours the introduction (or consolidation) of the commutative quality of additions.*

3. Starting from this activity, students are led to the **writing** of a number in an **additive** form, in various ways:

> The mini pyramid has a top brick that contains the number 12. Which numbers can be placed in the lower bricks. [3]



[3] *Students must find the pairs of numbers that make the sum of twelve:*

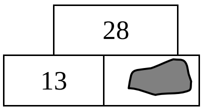
0 – 12	*12 - 0.*
1 – 11	*11 - 1*
2 – 10	*10 - 2*
3 – 9	*9 - 3*
4 – 8	*8 - 4*
5 – 7	*7 - 5*
6 - 6	

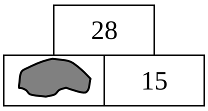
It would be wise to simulate an orderly research of the pairs of additions.

Suitable age related activities	6	7	8	9	10	11	12	13	Comments

4. After the additions and by means of suitable problems, one may begin with subtractions.

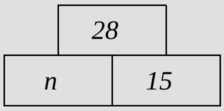
James has built his mini pyramid, Rita liked it and so she copied it. Unfortunately there is a **smudge** on Rita's mini pyramid, can you tell me the number that is under the smudge?





Expansion 1

With third and fourth year students who have already used a letter for representing an ***unknown*** *number, one may propose a representation which is different to the smudge problem number* **4:**



The next step consists in ***coding*** *the situation; this is very important metacognitive phase, because the students reflect on what they are doing. The request could be formulated like this: "****Translate*** *into* ***mathematical language*** *the way you would find the value of n".*
Normally a variety of different writings are found, for example:

- *28 – n = 13*
- *28 – 13 = n*
- *13 + n = 28*
- *n + 13 = 28*
- *n = 28 – 13*

Once the correct and incorrect writings have been written on the blackboard, they are then compared in a discussion (these aspects are described in depth in Unit 1: Brioshi and the approach to algebraic code).
Through this activity the teacher has the possibility of devolving on the students *the choice of how to represent the hidden number upon the students and of favouring – through the verbalization – the progressive social construction of meanings.*

Suitable age related activities	***6***	***7***	***8***	***9***	*10*	*11*	*12*	*13*	***Comments***

5. It is wise to get students used to the plurality of the representations of a number – that is, either the sum or the difference – for example: [4]

	27	
12		15

	12+15	
12		15

	38	
9		**29**

	38	
9		**38-9**

*[4] The use of non canonical representations helps in the learning of algebraic writings. It favours the **trasparency** of the **processes** and removes the emphasis on calculating. To evaluate this aspect and its evolution please consult the Situation 17 of this unit.*

Supplementary activities 1

One may also activate an itinerary, similar to the previous one but based on multiplications: the number of the upper brick is the product of the numbers represented in the lower bricks.

	18	
3		6

The activity continues by posing the problem:

> If we interchange the position of the numbers in the two lower bricks, does the number in the upper brick change or not?

3		6

6		3

Students are led to writing a number in a **multiplicative** form and *in different ways*, by proposing this problem:

> In the top brick of my mini pyramid there is the number 12.
> Can you tell which numbers there are in the lower bricks ?

	12	

Supplementary activities 1

The themes and their progressions are similar to the previous one:

- *Multiplication*
- *the language: factors, product*
- *the commutative property*
- *the opposite operation: divisions. It is opportune to insert the zero factor, too.*

This problem favours the introduction (or the consolidation) of the commutative property of multiplication

Students must find the multiplicative factors of 12 and their symmetries: 1-12; 2-6; 3-4

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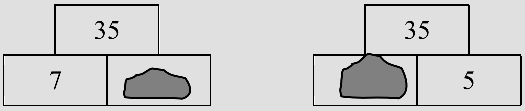
Suitable age related activities	6	7	**8**	**9**	**10**	11	12	13	*Comments*

Supplementary activities 2

The mini pyramids can also be used in presenting the *division* operations, by using *smudge problem* again.

Alice has built her mini pyramid and Adrian has copied it but there is a smudge on the paper. Can you tell me what was written under the smudge?



Student must get used to the *plurality of the representations* of the product. For example:

	56	
8		7

	8×7	
8		7

	56	
8		**56:8**

	56	
56:7		7

Supplementary activities 2

As in the **Situation 4.**, *it is opportune to present this smudge situation with the smudge on both sides, on the brick on the right and the brick on the left, so to prevent the stereotype that 'the* **result** *comes after the equal sign and it is always on the right'.*
On various occasions the letter 'm' is used to indicate indifferently or 'number under the smudge' or 'smudge'. For example: 35: 7 = m, 7 x m = 35.
Both aspects right/left and the smudge/number aspects will then be discussed with the class.

Second phase

Work now evolves towards pyramids with a three brick base; the same rule applies: the activity is notably enriched.
An example of a completed pyramid:

		42		
	18		24	
7		11		13

6. The first problem is presented to the pupils: [5]

Having the numbers in the bricks on the first floor, complete the pyramid.

10		5		19

Note 1

So as to prepare these situations, one must first construct a complete pyramid and then proceed by cancelling some numbers. Naturally the level of difficulty varies according to the students' age. The term 'floors' could be used, thus the base of the pyramid can be called 'first floor'.

[5] *It is important that the students verbalize their strategies by means of coding the processes activated. They are asked to represent for Brioshi their passages (the operations are the same but their order may change)*
1) 10 + 5 = 15
2) 5 + 19 = 24
3) 15 + 24 = 39

		39		
	15		*24*	
10		*5*		*19*

Suitable age related activities	6	7	**8**	**9**	**10**	11	12	13	Comments

Supplementary activities 3

While using the non canonical representations of numbers (as in situation number **5**) get students to reflect on the functionality of the brackets to indicate a process, however they should also be aware that the brackets – as in the following example of direct operations – may also be omitted thanks to the *associative law*.

	(9+8)+15	
9+8	15	
9	8	7

Problems are now proposed which require the use of subtractions [6].

[6] *It is fundamental that students understand how important it is for them to confront their strategies and to collectively reflect upon them. Verbalization is imperative so as to stimulate the metacognitive aspects and thus favours the control of the meanings.*

7. [7]

	22	
3	6	

7

1) 3 + 6 = 92) 22 – 9 = 13
3) 13 – 6 = 7

	22	
9	*13*	
3	*6*	*7*

8. [8]

	42	
	20	
8		

8.

1) 42 - 20 = 22
2) 22 – 8 = 14
3) 20 - 14 = 6

	42	
22	*20*	
8	*14*	*6*

9. This problem is different from those presented previously [9]

15	11	

[9] *It is a problem which can be solved in different ways, depending on if you start with – the upper brick or the lower brick. In the first case the choice of the first number is free, whereas in the second case the bigger number must be higher than 26 (the sum of 15 and 11).*

Suitable age related activities	6	7	8	9	**10**	**11**	12	13	Comments

Second phase

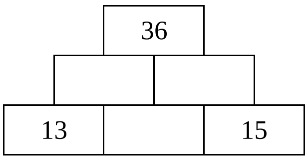
10. Another, more complex problem concludes the activity with the three-tier pyramids. The problem can be solved using two different strategies: **10**

- Arithmetically (by trial and error);
- Algebraically (setting up an equation).

The choice of strategy depends on the environment, and in particular whether the class has already encountered algebra, or the problem is to be used to approach it now.
The teacher can therefore choose one of several work hypotheses:

- To conclude (at least for the time being) the pyramids activity, working with small numbers, in order to allow for a solution by trial and error;
- To propose pyramids with bigger numbers, in order to make the search for the correct numbers difficult (too laborious), thus undermining the trials strategy, and start from scratch an algebraic solution, or retrieve existing knowledge, if the equations have already been tackled, e.g. using the unit From the Scales to Equations (see column next to ***Expansion***);
- To interrupt the activity, tackle the use of letters and go back - even after some time has elapsed - to solve the pyramid from a new perspective.

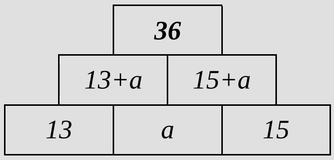
The following is a typical pyramid:



Some students get stuck and find this task impossible, however the majority proceed tentatively by placing a number in the central base brick and by finding the numbers of the second floor; the sum of which should be equal to 36.

Expansion 2

10 *If the students have already encountered the use of a letter instead of an unknown number, and in particular if they have worked on the unit U6: 'From the scales to the equation,' then the following problem may be the opening for the approach to algebra.*
One begins by leading the class to understanding that the problem could be solved if they knew the central base brick number. *If this number is represented by a letter (for example 'a'), then the two numbers on the second floor can represented in this way:*



The relation of equality which links 36 with 13 + a and with 15 + a leads to the equation:

$$13 + a + 15 + a = 36$$

the solution of which permits students to complete the pyramid:

$13 + a + 15 + a = 36$
$28 + 2a = 36$
$\cancel{28} - \cancel{28} + 2a = 36 - 28$
$2a = 8$
$2a : 2 = 8 : 2$
$a = 4$

Remember the delicate point of finding the solution to the equation are:

- *the transformation of the writing a + a in 2a;*
- *the opportune activation of additions (in this case the opposite of 28 from each side);*
- *The difference is nilled (28 – 28).*

It could be necessary to insist on the fact that the solution of the equation is the value that verifies the equality (that being 4) and not the number 36, as most students think, due to the stereotype that the result comes after the equals sign, on the right and thus is the ***result****.*

Suitable age related activities	6	7	8	**9**	**10**	**11**	**12**	13	*Comments*

Forth phase

11. The pyramid is made higher by adding another floor. The problem is always the same: how to complete the pyramid. An example of a completed pyramid:

68			
33	35		
17	16	19	
12	5	11	8

Some problems of increasing difficulty.

12. [11]

5	7	2	1

[11] Only additions are used.

33			
21	*12*		
12	*9*	*3*	
5	*7*	*2*	*1*

13. [12]

		9	
3	5		2

[12] The initial difficulty lies in understanding that to complete the base students have to subtract 2 from 9.

41			
20	*21*		
8	*12*	*9*	
3	*5*	*7*	*2*

1) 9 – 2 = 7 *2) 5 + 7 = 12*
3) 3 + 5 = 8 *4) 8 + 12 = 20*
5) 12 + 9 = 21 *6) 20 + 21 = 41*

14. [13]

10			
	6		
3			4

[13] The difficulties encrease. We 'start' with number 10 and we descend with three subtractions and we ascend with three additions.

25			
10	*15*		
4	*6*	*9*	
3	*1*	*5*	*4*

1) 10 – 6 = 4 *2) 4 – 3 = 1*
3) 6 – 1 = 5 *4) 5 + 4 = 9*
5) 9 + 6 = 15 *6) 10 + 15 = 25*

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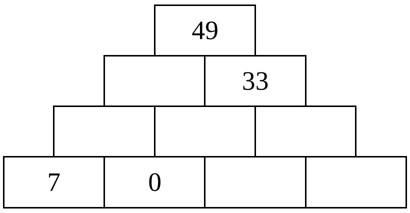
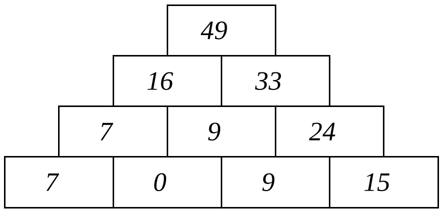
Suitable age related activities	6	7	8	**9**	**10**	**11**	**12**	13	Comments

15. [14]



[14] The processes in this case may be different but the operations are always the same. For example:

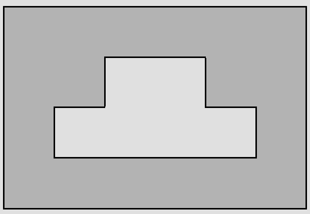
1) 49 –33 = 16	*1) 7 + 0 = 7*
2) 7 + 0 = 7	*2) 49 – 33 = 16*
3) 16 – 7 = 9	*3) 16 – 7 =9*
4) 9 - 0 = 9	*5) 33 - 9 = 24*
5) 33 - 9 = 24	*4) 9 – 0 = 9*
6) 24 - 9 = 15	*6) 24 - 9 = 15*



Note 2

The case may be, that some students do not realize how to activate the strategies to resolve these problems. To understand their difficulty one must reflect on what is the ‘right’ point of view: (i) one must ‘see’ the pyramids as *a set of mini pyramids* with various intersections and (ii) they can be resolved when one knows *two* numbers.

So as to favour this point of view it is helpful to use an artifice: a cardboard cut out which has an internal mini pyramid (in the paper modelling this kind of cardboard is called a ‘mask’). By placing this mask over the pyramid it is easier to observe the mini pyramids with two numbers.

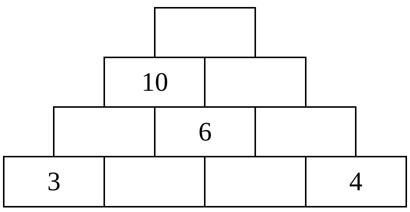


Note 2

We advise the teachers to look at previous pyramids using a mask or by means of a ‘mental mask’.
The mask was experimented in a first year secondary school, with a student who was affected by a Spastic Tetraplegia (Cerebral Palsy) and this method helped her to overcome her visual and manual difficulties in identifying the right mini pyramid within a context which had too many graphic elements that confused her.

It would be wise to conclude this section by presenting problems, which have been divided into different types.

A. Pyramid problems which can be resolved by using additions and subtractions.



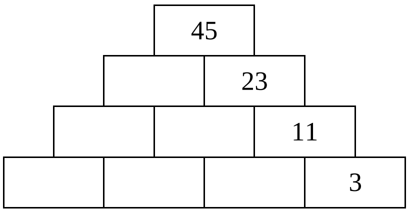
Examples of solution processes:

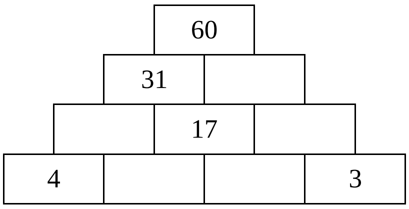
A.

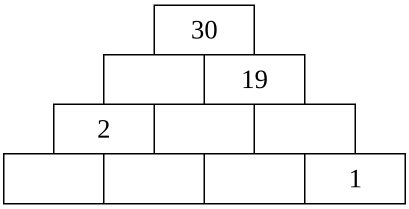
1) 10 – 6 = 4
2) 4 – 3 = 1
3) 6 – 1 = 5
4) 5 + 4 = 9
5) 6 + 9 = 15
6) 10 + 15 = 25

Suitable age related activities	*6*	*7*	*8*	***9***	***10***	***11***	***12***	*13*	***Comments***

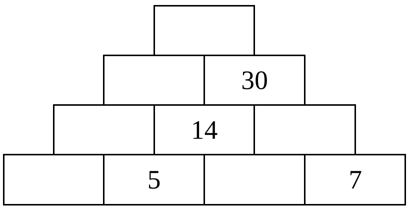
B. Pyramid problems which are resolved using only subtractions.



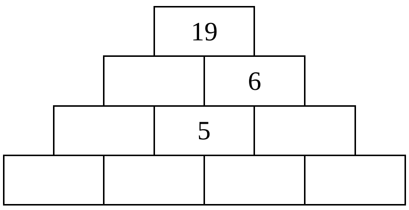




C. Open problems: partially undetermined pyramids.



D. Pyramid problems with base numbers that can be individualized by trials or by means of an orderly research.



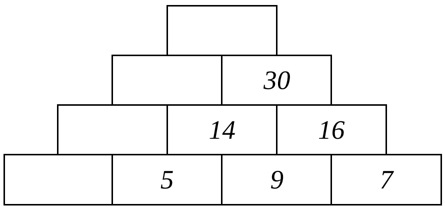
***B.** Examples of solutions:*

1) 45 - 23 = 22
2) 23 - 11 = 12
3) 22 - 12 = 10
4) 11 – 3 = 8
5) 12 - 8= 4
6) 10 - 4 = 6

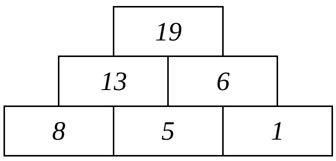
1) 60 - 31 = 29
2) 31 - 17 = 14
3) 29 - 17 = 12
4) 14 - 4 = 10
5) 17 - 10 = 7
6) 12 - 7 = 5

1) 30 - 19 = 11
2) 11 - 2 = 9
3) 19 - 9 = 10
4) 10 - 1 = 9
5) 9 - 9 = 0
6) 2 - 0 = 2

***C.** Once pupils have solved the 'fixed' bricks section, the remaining bricks section is solved by placing any one number in a brick and then consequently calculating the other three values.*



***D.** Having solved the 'fixed' bricks section, pupils must now find the base value, which must correspond to the upper bricks.*



There are two solutions:

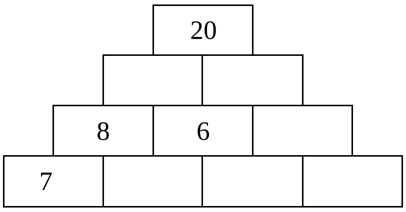
4	*4*	*1*	*0*

3	*5*	*0*	*1*

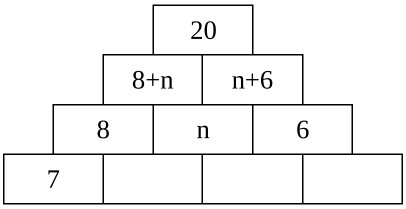
Suitable age related activities	6	7	8	9	**10**	**11**	**12**	13	Comments

Fifth phase

16. A new series of problems opens the way towards a passage to algebra.

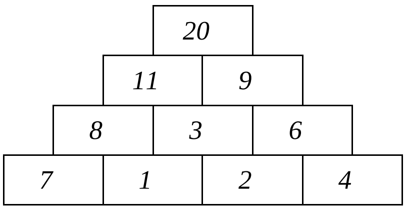


Students begin to complete the mini pyramid at the bottom on the left (8 – 7 = 1) but then they stop because some elements are missing for them to be able to continue.
If the numbers within the bricks are small (as in the example), and with some stimulation from the teacher (especially with younger students), pupils manage attempt a successfully strategy and thus complete the pyramid.
However, having bigger numbers, of course failure probabilities increment. This is the moment to propose the algebraic strategy and a letter is introduced, in much the same way as explained in **Expansion 2**; thus arriving at this representation:



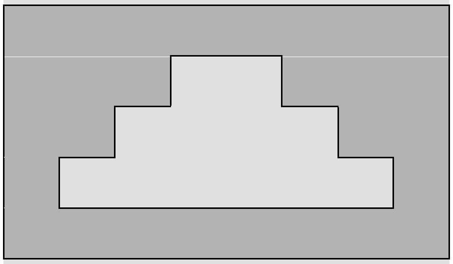
The next step and coding leads to the equation:
8 + n + n + 6 = 20
14 + 2n = 20
~~14~~ – ~~14~~ + 2n = 20 – 14
2n = 6
2n: 2 = 6: 2
n = 3

Thanks to the solution of the equation the pyramid is completed.

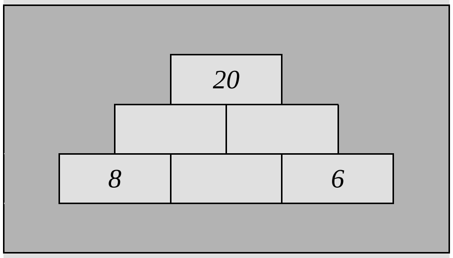


Note 3

The greatest difficulty consists in recognizing the three floor pyramid; yet again the ‘mask’ helps to identify which section they need to operate in.



The mask can be placed over the pyramid in three different positions and helps to understand that the algebraic strategy leads to identifying only one pyramid, if the numbers are positioned at the ‘top’ of a triangle.



Suitable age related activities	6	7	8	9	**10**	**11**	**12**	**13**	Comments

Note 4

Let us recall what was said in **Note 1** regarding the construction of a problem with pyramids: a complete pyramid is constructed and then some of its number are cancelled. Let us now see what we can obtain starting from the same pyramid: we can obtain two different types of problems with *arithmetic* solutions or *algebraic* solutions.

68			
33	35		
17	16	19	
12	5	11	8

A. ‘Arithmetic’ pyramids:

	8		
2		7	6

32			
3			
	1	7	

B ‘Algebraic’ pyramids:

11			
		13	
2		7	

Note 4

***A.** The solution in this case is simple …*

1) 8 - 7 = 1
2) 2 + 1 = 3
3) 6 + 7 = 13
4) 3 + 8 = 11
5) 8 + 13 = 21
6) 11 + 21 = 32

… whereas in this case it is more complex.

1) 3 - 1 = 2
2) 1 + 7 = 8
3) 3 + 8 = 11
4) 32 - 11 = 21
5) 21 - 8 = 13
6) 13 - 7 = 6

11			
2+a	*a+7*	*13*	
2	*a*	*7*	

The equation is formulated and solved:

$2 + a + 7 + a = 11$
$9 + 2a = 11$
$\cancel{9} + 2a = \cancel{9} + 2$
$2a = 2$
$2a : 2 = 2 : 2$
$a = 1$

Then pyramid is completed.

Project ArAl	U3: Numbers pyramids

Suitable age related activities	6	7	8	9	10	11	12	13	Comments

Sixth phase

17. An interesting algebraic development evolves from this problem:

> Represent with four letters the base numbers of the pyramid and by using them express the number in the top brick.

The class should organize step by step the following representation:

	a+3b+3c+d		
	a+2b+c	b+2c+d	
a+b	b+c	c+d	
a	b	c	d

Step by step, as the pyramid is completed, it is important that students reflect collectively and verbalize their **procedures**, in particular regarding the meaning of the final formula where it is possible to evidenciate the role of the laws of addition for having the passage to more accurate representations: *the number represented in the top brick has the same sum as the first number of the left base brick, plus the second number trebled, plus the third number trebled, plus the last number* [15].
Having done this, one continues with finding solutions to problems which are similar to the following:

18. [16]

> The numbers written on the base bricks, read from left to right are: 5, 1, 2, and 9.
> Can you find the value contained in the top brick without reconstructing all of the passage procedures.

[15] *Would be wise to dedicate particular attention to the fact that students must reflect on the writings equivalents such as:*

$$b + b = 2b \qquad c + 2c = 3c$$

which permit the transformation of the writings into more expressive ones. These are the necessary transformations to resolve the pyramids:

- $(a + b) + (b + c) =$
  $= a + (b+b) + c =$
  $= a + 2b + c$
- $(b + c) + (c + d) =$
  $= b + (c + c) + d =$
  $= b + 2c + d$
- $(a+2b+c) + (b+2c+d) =$
  $= a + 2b + b + c + 2c + d =$
  $= a + 3b + 3c + d.$

It is helpful to verify the formula on some numeric pyramids by resolving the same pyramid in two different ways.

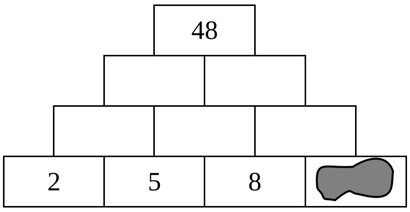
[16]**.** *This is a* substitution exercise *of numeric values in a literal expression; where the letter takes on an undetermined role. It is important that students pay attention to the underlined parts of the text, if this is not done they end up solving the problem in the usual way.*
The exercise rules must be respected so as to reason their way through to a conclusion of the situation ***16*** *which leads to this solution:*

$$5 + 3 \times 1 + 3 \times 2 + 9 = 23$$

Suitable age related activities	6	7	8	9	10	11	**12**	**13**	Comments

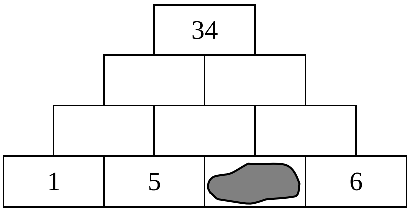
19. [17]

On the right base brick there is a smudge. Simon, who has completed the same pyramid, tells you that in the top brick there is the number 48.With this information, can you find the number under the smudge and complete the pyramid?



20. [18]

Yet again another smudge – let's see if you can complete this pyramid!



[17] *The student can start with either an arithmetic approach or an algebraic one.*

- *Arithmetic approach:*
  $48 - (2 + 3 \times 5 + 3 \times 8) =$
  $= 48 - (2 + 15 + 24) =$
  $= 48 - 41 =$
  $= 7$

- *Algebraic approach:*
  $2 + 3 \times 5 + 3 \times 8 + n = 48$
  $2 + 15 + 24 + n = 48$
  $41 + n = 48$
  $\cancel{41} + n = \cancel{41} + 7$
  $n = 7$

[18]. *Even this problem can be solved or with arithmetic or with an equation.*

- *Arithmetic approach:*
  $34 - (1 + 3 \times 3 + 6) =$
  $= 34 - (1 + 9 + 6) =$
  $= 34 - 16 =$
  $= 18$

The equation this time is more complex because the coefficient of the unknown number is 3, therefore the solution has an extra passage to work through.

$1 + 3 \times 5 + 3 \times n + 6 = 34$
$1 + 15 + 3n + 6 = 34$
$22 + 3n = 34$
$\cancel{22} - \cancel{22} + 3n = 34 - 22$
$3n = 12$
$3n : 3 = 12 : 3$
$n = 4$

Suitable age related activities	6	7	8	9	10	11	**12**	**13**	Comments

Expansion 3

21. An interesting development of **Situation 17** (that captures the student curiosity) can be found in the following problem: 19

> What will happen to the top formula if we add to the pyramid a new base with five bricks?

The class elaborates this pyramid:

a+4b+6c+4d+e
a+3b+3c+d | b+3c+3d+e
a+2b+c | b+2c+d | c+2d+e
a+b | b+c | c+d | d+e
a | b | c | d | e

A more amplified view is proposed:

> What will happen to the top formula if we add another brick to the base?

Their simple calculations, lead to this expression:

$$a + 5b + 10c + 10d + 5e + f$$

On a chart we write up the formulas related to all the pyramids we have worked on up to this moment:

Number of base bricks	formula
1	a
2	a + b
3	a + 2b + c
4	a + 3b + 3c + d
5	a + 4b + 6c + 4d + e
6	a + 5b + 10c + 10d + 5e + f

The class must reflect upon this increase:

> Without making any calculations can you find the formula which represents the top number of a pyramid which has a base of 7 bricks?

The transcription of the numeric coefficient of the monomials of each of the previous formulas leads to the understanding of the *Pascal Triangle* and helps in resolving the problem:

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

$$a + 6b + 15c + 20d + 15e + 6f + g$$

Expansion 3

***19*.**
This activity can be done collectively, individually or in small groups depending on the environmental conditions.

Project ArAl	U3: Numbers pyramids

Suitable age related activities	6	7	8	9	10	11	**12**	**13**	*Comments*

Expansion 4

Another aspect of this Unit is dedicated to the possibility of working with structures of *odd and even numbers*.
Some examples based on a pyramid with a four brick base:

22. [20]

> In a pyramid all the numbers of the first floor are *even* numbers. Can you tell me if you will find an odd or an even number at the top?

23. [21]

> What values must the base bricks have in order to have an *odd* number in the top brick?

24. [22]

> You have four mixed up bricks. In two of them there are even numbers and in the other two there are odd numbers.
> Position them on the base so that on the second floor you have *only even* numbers.
> Position them on the base so that on the second floor you have *only odd* numbers.

25. [23]

> If you swap over and change the bricks of the first line, will the top brick number remain the *same* or will it *change*?

26. [24]

> You have four bricks at your disposal for your pyramid and their values are: 2, 5, 8, and 11.
> 1. In what order must you place these bricks so as to have the *largest number* in the top brick?
> 2. In what order must you place these bricks so as to have the *smallest number* in the top brick?

Expansion 4

Even these problems favour discussions and collective verbalization, which aid the mental processes:

[20] *One may follow* two *strategies:*

- *Construct the pyramid and, by experimenting, discover that the second floor of bricks contains even numbers and thus conclude that the number at the top is even.*
- *Start from the known formula*
  a + 3b + 3c + d
  do the operations in (P,D,+) and (P,D, x) and conclude that the number is even.

[21] *The problem has* two solutions:
(i) three even values and one odd value, or
(ii) three odd values and one even value.

[22]

1) The first task does not *permit a solution.*
2) It is necessary to alternate *odd and even.*

[23]

Proceeding by trials student discover, much to their surprise, that by inverting the bricks, the top brick changes its value. *By analysing the formula, students realise that the central bricks are 'worth triple' (that is they have a coefficient of 3)*

[24]

As we have seen in the previous problem, the central bricks had triple value, whereas if we place the highest values of 8 and 11 we obtain the highest sum, and if we place them laterally we obtain the lowest sum.