"BRIOSHI" AND OTHER MEDIATION TOOLS EMPLOYED IN A RELATIONAL DIDACTIC APPROACH OF ARITHMETIC WITH THE AIM OF TEACHING ALGEBRA AS A LANGUAGE

Nicolina A. MALARA & Giancarlo NAVARRA
GREM Mathematics Department – University of Modena & Reggio Emilia -Italy

Summary
We shall trace the outline and the initial results of an experimental project on didactic innovation in primary schools aimed at the pre-algebra stage of maths teaching. The complex structure of this project entails the involvement of teachers in the discussion and reflection on the results of the experiment, making it at the same time a teacher training project. Here we shall concentrate on describing one of the newest and most productive teaching schemes based on "Brioshi", an imaginary Japanese boy used as the figure through which to study simple problems with the aim of representing them using communication via e-mail. The activity, carried out in third-year primary schools, is based on the analysis and discussion of the representation given by the pupils and on the negotiation of the most effective one which leads to their mastering the e-mail problems through the codification of simple processes and which activates the interpretation of connected algebraic expressions.

Introduction

Traditional teaching creates a divide between arithmetic and algebra. In arithmetic, the algorithms of calculus dominate and these arithmetic expressions are conceived as a sequence of operations to be carried out in order to obtain a certain result, while algebra concentrates on the study of symbolic representation (expressions, equations, functions) as the mathematical object. This passage from process to object, so typical in the development of mathematics (Sfard 1991,1994), is not sufficiently highlighted in teaching. The unspoken change of direction disorientates pupils and leads them to a *passive acceptance* of the rules and techniques of calculus *without control of the meanings* which these rules embody or of the properties on which these rules are based. Algebra taught in this way starts to lose several of its essential characteristics: on one hand, that of a language suitable for describing reality and on the other hand, that of a powerful tool of working out prediction through the formulation of knowledge (or hypotheses) of certain phenomena (in this case elementary) as well as the derivation of new knowledge about the phenomena through the transformations of algebraic formalism.

Many researchers (Bell 1987; Booker 1987; Kieran 1990, 1992; Kuchmann 1981) agree that the difficulties in the approach to algebra are rooted in the scarce attention paid to the relational or structural aspects of arithmetic which constitute the basis of elementary algebra. For example classic works from the 1970s (see Kieran's surveys, 1990, 1992) show the difficulty that beginners have in accepting the so-called "lack of closure" in expressions such as: 7+a, x+3, or more generally a+b, for the fact that in arithmetic, expressions such as 5+3 are not considered as representative of the number 8, but are seen as a procedure to carry out in order to obtain that number. Furthermore, numerous studies testify that even pupils who have no difficulty manipulating symbolic expressions are not able to translate relationships or tendencies observed into algebra, or to develop their reasoning in formal terms (see Bell 1976, Frielander *et al.* 1988, Kieran 1989, MacGregor 1991). We agree with these authors and believe, as shown also in our own experiments (Malara & Gherpelli 1997, Malara & Iaderosa 1999, Malara & Navarra 2000), that it would be suitable to extend the approach to these problems to the primary school level, with a syllabus co-ordinated between arithmetic and algebra, the latter considered as a language used to represent relationships and properties. This basic hypothesis is also shared by other researchers operating from various stances and with different approaches to the matter (Linckewski 1995, Ainley 1999; Da Rocha Falcao 1995, 1997, Carraher et al. 2000, Brawn e& Coles 2000, Sawadosky 1999).

Our specific hypotheses

On the linguistic level, some of the main difficulties that the younger pupils must face are shown by having to understand: i) *why* a symbolic language is used; ii) what conditions a symbolic language must adhere to; the difference between *solving* and *representing* a problem.

Let us start with the last point. The skill required for a task such as "How many are there …?" leads the student to concentrate on the result, and therefore on the calculation; the ability required lies at the cognitive level. On the other hand, a task such as "Explain how you did …" is more complicated because it is more difficult to examine oneself while calculating that to calculate. This ability lies at the "metacognitive" level.

For example, activities in which it is necessary to find the rules of a structure or in which students are required to translate from the spoken language to a symbolic one or vice versa are valuable in terms of algebraic thought training insofar as the pupil should learn to set aside the worry about the result and therefore also the implementation of operations that lead him or her to it. The pupil should reach a higher level of thought, replacing the "calculation" with "self examination during the calculation process". It is the passage to the metacognitive level at which the task- solver interprets the mathematical structure of the problem. Using such an approach, algebra becomes not only a language with which to describe reality, but it also widens ones understanding of it. However, a process of this type develops very slowly, step by step, through an interweaving mesh of continua and fragments strewn between different levels of consciousness.

Let us now look at language. We believe that there exists a strong analogy between the methods of learning spoken language and that of the algebraic language. In order to explain this point of view, we shall employ the metaphor of the *stutter*.

When a child learns a language, he or she masters the meanings of words and their supporting rules little by little, developing gradually by imitation and self-correction right up to the study of the language at school age at which the child starts to learn to read and reflect on the grammatical and syntactic aspects of the language. Traditionally, in the teaching of the algebraic language, one starts with the study of the rules, as if the formal manipulation should have precedence over the understanding of meanings. The syntax of algebra therefore tends to be taught while overlooking its semantics. The mental models of algebraic thought should rather more be built around that which we might call the initial forms of the *algebraic stutter*, starting right from the first year of primary school, from the moment in which the child first starts to approach arithmetic thought. In other words, it is necessary to teach the pupil *to think of arithmetic in algebraic terms*. However, traditionally algebraic thought is not built *progressively* as an instrument and object of thought *in parallel* with arithmetic, but only afterwards, making it impossible to construct an environment which might stimulate even informally the autonomous elaboration of the stutter and therefore the playful, experimental and continuously redefined acquisition of a new language.

Let us consider the following example. For a problem like this: *On one branch there are 13 crows; on another there are 6.* We might imagine two types of task: a) *Calculate the total number of crows*; b) *Explain how you might find the total number of crows and then calculate it.*

The first task favours a mental approach which leads to the immediate search for the tools (the operations) needed to identify *the answer* (the result). This result can be associated with the primary needs that a young child tends to try and satisfy in his or her initial use of language (food, sleep, pleasure etc.). The second task, on the other hand, stimulates reflections on the part of the child performing the calculation (even in very small children) together with reasoning abilities which are anything but babyish. Basically, a strong development of arithmetic thought almost entirely focused on operations with familiar numbers can cause the formation of stereotypes in the pupil's mind which in the long run become impossible to erase. Because of this, the student gets trapped in the obsessive search for a numeric result (*the primary need*), which thus prevents him or her from exploring alternative mental channels which might be more fruitful and stimulating in regard to the formation of the first true algebraic thought, aimed at the *interpretation* and *description of reality* through such a language.

The prospect of starting off the students with algebra as a language, continuously thinking back and forth from algebra to arithmetic, may encourage the individualisation of a more effective syllabus with pupils of between 7 and 14 years old based on the *negotiation* and then on the *explicitation* of a didactical contract for algebraic problems in order to find the solution based on the principle "*first represent, then solve*". This prospect seems very promising when facing one of the most important issues in the field of conceptual algebra: the transposition in terms of *representation* from the spoken language in which the problems are formulated or described to the formal algebraic language into which the relationships are translated. In this way the search for the solution is part of the successive phase.

Like every language, the language of mathematics also possesses its own grammar, or rather an ensemble of conventions which allows us to build "sentences" correctly. It has its own syntax which provide us with the conditions – the rules – needed to establish whether or not a series of linguistic elements is "properly formed" (for example, sentences such as these are incorrect: "9 + + 6 = 15" or 5 + 3 = 8 : 2 = 4 + 16 = 20"). It has its own semantics which allows us to interpret symbols (within syntactically correct sequences) and subsequently establish whether or not the expressions are true or false (for example the sentence "1 + 1 = 10" may be true or false depending on the basis of calculus used: it is false according to the system based on 10, but true in that based on 2).

As regards which of the two analyses – the semantic or syntactic one – must precede the other, we maintain that *the second must follow the first*. The implications of this statement are very relevant.

From this point of view, *translating* sentences from spoken (or graphic or iconic) language to mathematical and vice versa represents one of the most fertile areas within which reflections on the language of mathematics may be developed. Translating in this sense means *interpreting* and *representing* a problematic situation through a formalised language or, conversely, recognise a situation described in symbolic form.

Our own specific hypothesis is that the mental frameworks of algebraic thought should be built right from the earliest years of primary school when the child starts to approach arithmetic thought by teaching him or *her to think of arithmetic in algebraic terms*. In other words, this means constructing algebraic thought in the pupil progressively as a tool and object of thought working in parallel with arithmetic. It means starting with its *meanings,* through the construction of an environment which might informally stimulate the autonomous elaboration of that above mentioned *algebraic stutter*, and then the experimental and continuously redefined mastering of a new language in which the *rules* may find their place just as gradually within a teaching situation tolerant towards the initial, syntactically "shaky" moments which stimulates a sensitive awareness towards these aspects of the mathematical language.

Brioshi the imaginary pupil as an element of mediation in the approach to algebraic representation

This scheme also includes the *Progetto Speciale ArAl – percorsi nell'aritmentica per favorire il pensiero pre-algebrico* (ArAl Special Project – courses in arithmetic designed to encourage pre-algebraic thought) which our research group set up in 1997 on these issues in primary schools[1]. Nine schools are involved in the ArAl project, with 66 teachers and almost 1,500 pupils (1,200 in primary schools and 300 in middle schools). An aspect of great importance is represented in the search for a link between the activities presented by the project and the mathematical curricula for junior schools, with the objective of individualising the ways of obtaining a progressive integration between the two. This aspect is heart-felt by the teachers because, if on one hand it reflects the widespread fear of having to "*make space for algebra"* within a program which is considered ample, on the other hand it also represents the opportunity *for a reflection of ones own*

[1] ArAl is the continuation of a previous project set up in 1993 on these matters but aimed at middle schools (Malara 1995; Malara & Iderosa 1999). This constitutes one of the Italian contributions to the international ELTMAPS project run by L. Rogers in which Great Britain (Faculty of Education, Roehampton Institute, London), Cyprus (Pedagogical Institute, Nicosia), the Czech Republic (Mathematics Department, Pedagogical Institute, Charles University, Prague) and Italy (Mathematics Department, University of Modena & Reggio Emilia) participate.

knowledge and convictions in the mathematics field so as to arrive at an objective critic of the readings, contents, methods and strategies

A key element in the development of the project is the use of a hypothetical Japanese boy Brioshi: Brioshi is a "virtual" student, of variable age, according to the age of his interlocutor. He doesn't speak the Italian language but he knows how to express himself using a correct mathematical language. Brioshi's Japanese class loves meeting other classes of the same age who are not Japanese, so as to exchange mathematics problems via email. Of course the messages, especially with younger students, can also contain sentences written in Italian or in Japanese (" Dear Brioshi, I want to know if you are capable of…..") which accomplishes – due to the total lack of understanding for the receiver – a marginal function from the ordinary language point of view but which make it powerful and significant in its universality, the use of the arithmetic algebraic code, the heart of the message (represented by the dots) is the mathematical nucleus of the speech. Brioshi has been introduced to all of the classes involved in the project, and he provides a powerful support which often helps to transmit otherwise difficult concepts of understanding for students between the ages of 8 and 14: the need to respect the rules of the language, as we hinted at in the second of the three initial points. That is to say, the need to respect the rules in the use of a language, which is an even stronger need when this has to be formalised, is the motive behind the extremely synthetic nature of the symbols used. It is therefore very fruitful to make use of Brioshi whenever the pupil finds him or herself faced with the question of representation and translation.

Teaching situations involving Brioshi

Teaching situations involving Brioshi have been developed within the ArAl framework and are now in fact starting to constitute a sub-project. The teacher starts by proposing a message swap between pupils beginning with very simple sentences in Italian which the pupils try to translate into mathematical language. Each time, the various translations are put up together on the board and are discussed collectively so as to chose the one to send to Brioshi. Once the translation has been sent, the class waits for Brioshi's answer and it is interpreted. Likewise, Brioshi's class sends problems to the class which must be interpreted and answered. The exchange with Brioshi may be stimulated within the class itself and in this case it is then the teacher who proposes the "reply message" or who invites one of the students to come up with one.

This "role-play" always works, regardless of the age of the pupils. The exchange becomes even more interesting and effective if it takes place with another class through the exchange of slips of paper. E-mail messages brought in by the teacher are also very effective. They arouse curiosity in the students and present a logical/mathematical challenge. The most interesting methods – put into practice in January 2001 – are to do with a message swap between Brioshi classes involved in "mathematical communication" in real time in a special internet chat-room.

The analysis of the pupils' overall output in the project indicates analogies and differences between items which are usually anything but banal, and has allowed us to organise an interpretation grid aimed at interpreting the translations for two different purposes: firstly, for the researcher for use within the research project, and secondly, for the teacher to sharpen his or her sense of the linguistic aspects of mathematics. Consequently, both the organisational criteria of the grid and the terminology used help to consolidate a concept of mathematics from a linguistic prospective. The pupils' output can be classified in three categories. Each category covers various typologies of output which we have classified as shown in the grid in Table 1.

Table 1

Categories of classification and related codes
1 The translation shows a good understanding of the situation.

L. Literal: use of the letter (translation faithful to the text)
The pupil (aged eight) understands both the problem and the task. The use of the letter is explained with phrases like "p stands for the place of the number I don't know", or "n represents the missing number". The translation is always relational; sometimes the output is correct but "excessive" compared to the text (Le); it may show the inversion of the addenda or factors; it may be "free" (Ll) (the pupil sees the correct relationship between the numbers; the translation indicates an elaboration of the verbal message. For example:
F Faithful: use of symbolic alternatives to letters (translation faithful to the text)
The pupil understands the logical structure of the problem but exercises partial metacognitive control over the meaning of the symbols used; sometimes the output is correct but "excessive" compared to the text (Fe, second example); it is often written "in columns" denoting a fundamentally operative attitude (O, see last example).
S Sense: a clearly operatively based attitude (translation faithful to the text)
The pupil shows a substantial understanding of the logical structure of the problem but exercises only partial metacognitive control; the translation "pre-empts the result" and puts down a directional rather than relational letter.

2 The translation shows a partial understanding of the situation.
C Confused: partial adherence to the text
The pupil uses, co-ordinates or elaborates the initial data wrongly; the language used is often mixed (spoken/symbolic).

3 The translation shows an insufficient understanding of the situation.
I Unfaithful: lack of adherence to the text
The pupil puts together the initial data wrongly (often with an erroneous operation).
A Adulterated: Total lack of adherence to the text
The pupil shows lack of comprehension at all levels (often due to a misunderstanding). In the case of more articulated problems, this may be shown by an iconic representation of the problem to be solved which is sometimes faithful to the text yet in some way "blocked".

Transversal caracteristics
N solving number: insertion of the value of the unknown (poor adherence to the text)
The student partly understands the situation and exercises weak metacognitive control; he or she carries out the operation without realising that the writing does not constitute a "problem" (short-circuiting of the problem).
Operational: insertion of the "=" (adherence to the text), may appear in any type of output. The pupil shows an attitude aimed dominantly at "calculating" rather than "representing". The absence of the "=" is seen as a "lack of closure". The output may be either in lines or in columns.

Analysis of diaries relative to several types of activity

We shall now analyse (through a synthesis of diaries relative to the activities) some of the situations which presented themselves in primary school classes (eight-year old pupils) which have adopted the Brioshi project. The classification codes are given next to each translation since a translation may have more than one characteristic. The key to reading the diaries is: 🗪 group discussion; 🗩 ensemble of various interventions; 🗨 intervention of a single pupil; ✓ intervention of the researcher.

1st situation (pupils in the third month of participation in this project)

Translate this sentence for Brioshi: "Take 8 away from 15".
Translations put up onto the board and discussed with class
L: (a) 15 – 8; LR: (b) 15 – 8 = 7 ("in excess"); LO: (c) 15 – 8 =
🗪 The first trials highlight the difficulties found in comparing and interpreting the writings; the analogies between writings are not grasped.
🗩 Most of the children evaluate positively the insertion of the "=" symbol. (a) is considered incomplete.
🗨 Lucia observes that the most correct way to translate the sentence in her opinion is (a) (her own) because they weren't asked to find the answer but to translate the sentence in a way that Brioshi might understand!.
🗪 The observation is not understood by many of her classmates. The class is not sure which to choose.
✓ We ask them to write the number 4 in different ways.
🗩 Proposals: 1 + 3, 2 + 2, 5 – 1, 20 – 16, 2 x 2, 4 x 1, 100 – 96.
✓ We ask why none of them wrote "=".

Because you asked us something that came to 4!”.
You asked us other ways to do it, not another operation”.

There are two relevant aspects of this diary (which are also highlighted in the materials used by the other teachers in the experiment): (a) the “equals” sign which appears in two writings is felt by most of the students as a necessary indicator of conclusion, and expresses an expectation that a conclusion must in some way be reached. It denotes the prevalence of an underlying operative attitude. The absence of the sign, on the other hand, is seen as a lack of closure in the operation (in fact, the class considers (a) “incomplete”). Lucia’s observation introduces the explanation of why the "equals” sign is not necessary, and opens the way to pinpointing the difference between resolving and representing a problem. Further on, as often happens, the students ask “if you have to write down the answer”. In order to try and clarify this distinction between “translating” and “solving” a problem, we make use of a translation from Italian to English. We ask the pupils how they would translate the sentence *“Hai una matita?”*. The class answers, “Have you got a pencil?”. The child we ask this question to has a pencil in his hand and answers, “Yes”. In the first case the answer is translated, while in the second it is given an answer.

2nd situation (pupils in their fourth month of participation in the project)

Translate these sentences for Brioshi: Add four to an unknown number to get ten.
Student translations written up on the board and discussed:

F:	(a) + 4 = 10 ;	(b) 4 + = 10 ;	(c) 10 – 4	
FO:	(d) + 4 = 10 =			
I:	(e) 4 E 10	(f) 4 = 10		
IN:	(g) 6 + 10			
N:	h) 6 + 4 = 10 ;	(i) 6 + 10		
A:	(i) 1 + 13 ;	(j) 3 + 10 ;	(k) 3 + 7 ;	(l) 10 x 10

The class chooses (a) almost unanimously “because the empty space shows Brioshi that you have to find out the missing number”.
✓ We point out that the computer can’t leave an empty space and so we ask them what we can do.
Proposals: (m) p + 4 = 10 ; (n) ? + 4 = 10 ; (o) @ + 4 = 10.
The girl who proposed (m) explains that p stands for “the place of the number”
The class decides to send (m).

Writings (a) and (b) give the chance to make some interesting observations. They are equivalent on the mathematical level (it is right to underline the commutative properties), but it is (a) more than the other that respects the sequential aspect implicit in the task. It is interesting how (like in (m)) the letter ‘p’ appears, if only at the second stage. It is very rare for a letter to be offered up spontaneously like in this case. Often it comes down to a question of children having seen older brothers and sisters using letters. The choice of symbols must be very free; in particular, the use of the letter should be the outcome of collective negotiation and therefore depend on the class environment. For example, at the beginning of the activity, the empty space is the most commonly “plumped for” because it seems to communicate most clearly “that Brioshi is supposed to come up with a number”, or the decision of the class as regards the translation to send goes for an iconic sentence rather than a literal (more advanced) one. We let the discussions carry on highlighting the similarities and differences between writings. These aspects are very important in building semantically meaningful foundations in the construction of the algebraic language. As we shall see, it is the gradual refinement of the evaluations which leads to the most accurate identification of the correct writings.

3rd situation (pupils in their fifth month of participation in the project)

1st Problem situation*: Invent a problem which can be translated for Brioshi with the sentence: 30 – n = 6.*
Translations offered by the pupils:
(a) “I’ve got 30 sweets. I take away n sweets and I have 6.”
! They are asked to improve the wording of the text.
(b) “I’ve got 30 sweets. I take away the missing number and I have 6.”
We ask them to avoid using “mathsy” words (they smile).
(c) “I’ve got 30 sweets. I eat some and I have 6.”
(d) “I’ve got 30 sweets. I eat some and 6 are left.”

We ask them what other words could replace "some".
(e) "I've got 30 sweets. I eat a few and 6 are left."
(f) "I've got 30 sweets. I eat several and 6 are left."
2nd problem situation: *Invent a problem which can be translated for Brioshi with the sentence: 24 : n = 4. You mustn't talk about sweets anymore, but about hamsters.*
Translations offered by pupils:
(g) "I've got 24 hamsters. I split them up…"
(h) "I've got 24 hamsters and some children. I give four hamsters…"
(i) "I've got 24 hamsters. I split them up a bit and I've got 4 hamsters."
(j) "I've got 24 hamsters and some children and I give four of them…"
(k) "I've got 24 hamsters. I split them up in equal parts by four…"
Finally one girl clicks and unblocks the situation:
(l) "I've got 24 hamsters and I put them in cages and four can go in each cage. How many cages do I need?"

The translation from the mathematical language to the spoken one creates greater difficulties than translating the other way. Even the youngest students prefer the reflection on and verbalisation of mathematical writings. The introduction of indefinite adjectives can lead to very fruitful linguistic reflections, highlighting the equivalence between terms such as "a bit", "a few", "some", "several", etc. which all correspond to the meaning of "a part of…".
Though more difficult (division is an operation which is very new to the class), the second situation presented in the diary allows the class to clear up their numerous doubts. (i) and (j) introduce a partition division, (h) one of belonging; the missing number is difficult to handle and obstructs a wider view of the situation. The pupils try to get round the obstacle by introducing general words such as "a few" and "I split them up a bit". (k) introduces the "division in equal parts", but links it rather feebly to the number 4. (l) solves the situation brilliantly by introducing the cages and proposing a division in parts.
This type of activity is carried out in other contexts as well (geometric, the family, buying and selling, etc.).

Some notes on the attitude of the teachers who took part in the project

The majority of Italian primary school teachers are given no formal maths training. The ArAl project therefore constitutes an important opportunity for the teachers taking part in it to reflect on their own knowledge (which is, after all, the most important factor when they decide how to pass on knowledge to their own pupils) and their own convictions in the field of mathematics – one might even say of their own epistemology – in order to come to a critical vantage point regarding contents, methods and strategies.
Brioshi represents a very stimulating didactic environment from this point of view, though it is not always easy to handle. For example, a pupil in 4th year primary school sends a message like the following: "Dear Brioshi, let's see if you can solve this problem: 33 = a – 26". Brioshi might give an answer formulated from an arithmetic point of view (33 = 59 – 26), but he might also give a more complex answer, formulated from an algebraic point of view, such as: a = 59 or a = 33 + 26. It is clear that every time it comes, Brioshi's answer itself becomes an interpretational problem for the class that has sent the message and opens up new pathways to be explored. If however, Brioshi were to answer a question wrongly, such as a = 60, the class might be stimulated to elaborate an answer such as a ≠ 60.
The organisation of activities therefore entails various delicate aspects requiring numerous abilities on the part of the teacher. Among these, we might include (i) carrying out a sharp analysis of the pupils' proposals to discuss as a class (and in limited periods of time), (ii) classifying the majority of proposals (often written using a variety of personal language and symbols, not always in the most suitable way); (iii) knowing how to identify (and make the pupils identify) the paraphrases of a possibly correct translation (for example, teachers are often undecided as to whether or not to accept a writing with an addition when their mental model of a situation referred to a subtraction as right. This happens when a task like "Translate this sentence into mathematical language: "By how much is 7 less than 13?" is translated by the pupils both as "13 – 7 = 6" and "7 + 6 = 13"); (iv) checking the conceptual aspects tied to the different denotations of a mathematical object, when each also has a profoundly different sense. For example, uncertainties often emerge when faced

with the evaluation of equivalent denotations such as “14: 7 + 2” and “4” without grasping the fact that the second (which in relation to the first is seen as the “result”) is opaque because in the canonical form, we lose the aspect of the process while that of the product dominates.

The work with Brioshi is therefore important because it helps not only the student but also the teacher to understand that every task in the field of mathematics is open to a multi-level interpretation, including the way its formulation can be organised in terms of spoken language.

References

Ainley J., 1999, Doing algebra type stuff: emergent algebra in the primary school, *proc. PME 23*, v. 2, 9-16

Bell A., 1976, A study of pupils' proof explanations, *Educational Studies in Mathematics*, v. 7, 23-40

Booker G., 1987, Conceptual obstacles to the development of algebraic thinking, *proc PME 11*, v.1, 275-281

Brown L., Coles A., 1999, Needing to use Algebras – A case Study, *proc. PME 23*, v. 2, 153-160

Frielander A., Hershkovitz R., Arcavi A., 1988, Incipient "algebraic" thinking in pre-algebra students, *proc. PME 12*, v. 1, 283-290

Kieran, K.: 1990, Cognitive Processes involved in Learning School Algebra, in Nesher P. & Kilpatrik J. (eds), *Mathematics and Cognition*, ICMI Study Series, Cambridge University Press, 96-112

Linchevski, L.: 1995, ‘Algebra with numbers and arithmetic with letters: a definition of pre-algebra’, *The Journal of Mathematical Behaviour*, v. 14, 113 - 120.

Kuchemann, D.E.: 1981, Algebra, in Hart K. (ed) *Children Understanding Mathematics: 11-16*, Murray, London

Mac Gregor, M.: 1991, *Making Sense of Algebra, Cognitive Processes Influencing Comprehension*, Deakin University press, Geelong, Victoria, Australia

Mac Gregor, M. & Price, E.: 1999, ‘An exploration of aspects of language proficiency and algebra learning’, *Journal for Research in Mathematics Education*, v. 30 (4), 449 - 467.

Malara, N. A.: 1995, Mediating Theory and Practice: a case study, in Bazzini L. (ed) *Theory and Practice in Mathematics Education*, ISDAF, Pavia, 157-169

Malara, N. A., Gherpelli, L.: 1996, Argumentation and proof in the Elementary Number Theory in the triennium of middle school, *L'Educazione Matematica*, anno XVIII, serie V, v. 2, n. 2, 82-102

Malara, N.A., Iaderosa, R.: 1999, Theory and Practice: a case of fruitful relationship for the Renewal of the Teaching and Learning of Algebra, in Jaquet F. (ed) *Proc CIEAEM 50 -Relationship between Classroom Practice and Research in Mathematics Education*, 38-54

Malara, N.A., Navarra G.: 2000, Explorative ways to encourage algebraic thinking through problems, in Gagatsis, E, Makrides G. (eds) *proc. II MEDCONF on Mathematics Education*, v. 1, 55-64

Rocha Falcao J.T.: 1995, A case of algebraic scaffolding: from balance scale to algebraic notation, *proc. PME* 19, v. 2, 66-73.

Brito Lima A.P., Da Rocha Falcão J.T.: 1997, Early development of algebraic representation among 6-13 year-old children: the importance od didactic contract, *proc PME 21*, v. 2, 201-208

Savadosky P.: 1999, Arithmetic and algebraic practices: possible bridge between them, vol. 4, 145-152

Sfard, A.: 1991, On the Dual Nature of Mathematical Conceptions: Reflections on Processes and Objects as Different Sides of the Same Coin, *Educational Studies in Mathematics*, v. 22, 1-36