Unit 5

Regularities: frames and necklaces

1st – 8th grade

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U5. Regularities: frames and necklaces

1. The Unit

The activity on **regularity** was born within the ArAl project during the school year 2000-2001, drawing inspiration by some activities proposed in *The National Numeracy Strategy* of the Department for Education and Employment in Great Britain. The activity has gained a role of growing relevance among the works of the project and turned out to be very important mainly in favouring the **processes** of abstraction and generalisation.
The Unit treats the search for regularity in *frames* and in *pearl necklaces*. It represents only the first part of the experimental activity carried out in the classes of the project. The second part, regarding the search for regularity in a more precisely mathematical context, will be treated in Unit 7: *Regularity: the arithmetical sequences.* Studying frames and necklaces (as well as repeated groups of letters, or regular dispositions of toothpicks) represents one of the possible approaches to use, in order to introduce younger pupils to subjects connected with the search for regularities, because it is based on the use of real objects or of drawings. This allows the *perception* of the explored situation and hence favours the *intuition* of regularities, which will undergo, later on, reflection and logic elaboration.

2. Didactic aspects

The Unit starts off with the observation of regularities in frames, in repeated strings of letters, in sequences of pearls in various necklaces. This work is very suitable for a **collective discussion,** like an ideal training in order to favour the building of knowledge, and it gets to a first level of formalisation through the 'writing of formulas' of the regularities that have been detected. Within this Unit, there is a solid work on **argumentation**: pupils are asked to justify their intuitions, sometimes disproved by their classmates, with arguments that are often fairly sophisticated. In this Unit as well, we use **Brioshi** (see Unit 1*:* Brioshi and the approach to algebraic code) to give a motivation in the use of algebraic language.

3. General aspects

- Since its first phases, the activity is divided into problems; the class – divided into groups or through individual activities - explores situations of growing complexity and tries to solve them. The **verbalisation** and collective comparison of the adopted strategies allow to spread and consolidate the results of the *discoveries* as they are made.
- The Unit is fit to be developed in the last two years of primary school and in the first year of intermediate school.
- Collective discussions are of fundamental importance because they force to reflection on mental processes, to verbalisation of their own thoughts and their strategies, to listen to the others, thus helping to heighten not only the cognitive aspects, but also the meta-cognitive and meta-linguistic ones.
- The Unit offers various hints in different directions, above all in the algebraic field (reflections on division, on its remainder, on modular arithmetic).

4. Terminology and symbols

Phase	Sequence of situations of growing difficulty referred to the same subject.
Situation	Problem around which individual, group or class activities are developed.
Expansion	Hypothesis of work on a possible expansion of the activity towards an algebraic direction. Its realisation depends on the environmental conditions and on the teacher's objectives.
Supplementary activity	Enlargement towards subjects related to those developed in the preceding **Situations**.
Note	Methodological or operational suggestions for the teacher.
	In the square a problematic situation is proposed. The text is purely indicative; it can also be presented as it is, but generally its formulation represents the outcome of a **social mediation** between the teacher and the class
Representation	An underlined word in boldface type highlights a link to a subject illustrated in the Glossary.

Square containing the outline of a typical discussion; the following symbols may appear:

√	Intervention of the teacher
	Intervention of a pupil
	Summary of several interventions
	Summary of a collective discussion (a principle, a rule, a conclusion, an observation, …)

5. Phases, situations and subjects

All the phases are abundantly supported by significant extracts from the **Diaries** of activities in the classes.

PHASES	SITUATIONS	SUBJECTS
First	**1, 2**	Analysis of sequences: frames, letters and words; discovery of the 'stamp'
Second	**3**	Correspondence between the position of a pearl in necklace and its colour, resort to the stamp
Third	**4**	Correspondence between a pearl of a given colour and its position in the necklace; patterns and sub-patterns
Fourth	**5, 6, 7**	Necklace formed by alternated couples of pearls of the same colour: search for the colour of the pearl in function of the position in the necklace in the pattern and sub-pattern
Fifth	**8**	Necklace formed by alternated couples of pearls of the same colour: search for the position of the pearl in function of the colour

6. Distribution of the situations in relation to the age of the pupils

The unit can be proposed in its entirety to the fourth and fifth year of primary school and in the first year of intermediate school, without any significant variation either from the point of view of the contents, or that of times.
Although it has not been experimented in the third years (primary school), we think that the First phase might also be performed with pupils of this class, and might thus represent a possible background for deeper studies in the following years.

			PHASES AND SITUATIONS							
			I		**II**	**III**	**IV**			**V**
School	Age	Class	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
Primary	**8**	**3**	10 hours							
	9	**4**	15 hours							
	10	**5**	10 – 15 hours							
Intermediate	**11**	**1**	10 hours							

Suitable age related activities	6	7	***8***	***9***	***10***	***11***	12	13	*Comments*

Note 1

The activity starts with the study of a real pearl necklace in which 2 grey pearls are alternated with 5 black pearls. In order to suggest the concept of ‘necklace of unknown length’, we can recur to the trick of holding the closed necklace in our fist and letting the pupils see only the initial twenty pearls; thus we'll define the first two pearls as the ‘beginning of the necklace’.

The first work phase will consist in the identification of the colour of the pearls in a defined position, gradually increasing the ordinal number of the position. For example:

- What's the colour of the 12^{th} pearl?
- What's the colour of the 35^{th} pearl?
- What's the colour of the 123^{rd} pearl?

As long as we work with visible pearls, inevitably children tend to refer to the necklace and count the pearls. For example, they can easily see that the 12^{th} pearl is black. But when numbers increase, it is necessary to find a strategy, because the 35^{th} pearl is ‘hidden’ within the fist. It isn't easy to find a strategy either for the pupils, or - often - for the teacher him/herself, who is not used to working in conditions of search for regularities.
The basic difficulty consists in the *initial perception* we have of the necklace: the spontaneous one usually grasps *the alternation* of groups of pearls in different number and colour, and the search for the colour of a hidden pearl gets lost in this endless repetition. Some regularities are grasped but, so to speak, they are ‘fragmented’ in an unproductive way among grey and black pearls, between number 2 and number 5. This fragmentation blocks the development of the exploration and hinders the identification of the **structure** of the necklace, that is in the net of **relations** existing among the pearls, which form it. The identification of a regularity, in fact, derives form a *break* with perception, it is the result of a *meta-cognitive* operation: it is necessary to elaborate some information given by perception *totally changing one's point of view*. Gestalt psychology would define this as a necessary *reorganisation of the field*. Therefore, this is what we aim at in the **First phase**, preparatory to the **Second phase** in which we'll start working with a real necklace.

First phase

The activities proposed in this phase are optional and depend on the environmental conditions in which they are developed (is the teacher expert? is the class used to ‘exploring’ problematic situations? have experiences in the identifications of regularities already been faced?). If the answer to these questions is ‘no’, we suggest to do them. We suggest to read them anyway, also as a strategy to be added to the activity in cases when some pupils cannot productively treat the exploration of regularities.

ArAl Project	U5. Regularities: frames and necklaces

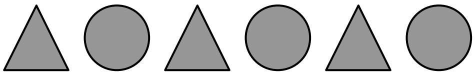
Suitable age related activities	6	7	8	9	10	11	12	13	Comments

1. Decorations and stencils
Stencils are instruments used since pre-primary school for the creation of frames, decorations, and so on. They perfectly fit our intentions because they help focus the pupils' attention on the *change in the point of view* we referred to in Note 1.

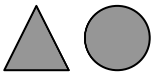
1 In maths they are called mono directional frames.

Diary 1 (fourth year, primary school, 9 year olds, October)

√ We propose a frame and ask how we might build a stamp to realise it.



«I'd use two stamps, one with the shape of a triangle and the other with the shape of a circle»
√ «And if we want to use a single stamp?»
The discussion leads to this solution:

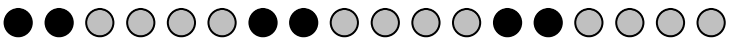


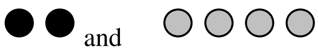
√ «In this frame how can we know what symbol is at the15th place?»
(The same little girl as before) «All the evens are circles and all the odds are triangles. Thus, there is a triangle at the 15th place»
The pupils immediately understand the meaning of their classmate's conclusion. Other similar questions arise no difficulty.

Diary 2 (fourth year, primary school, 9 year olds, October)

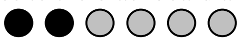
This diary shows how the reorganisation of the field is everything but a painless process, but how in the end it turns out to be very useful, most of all when it represents a social achievement, the result of personal elaborations, comparisons, discussions, common choices.

✓ We propose this decoration and ask to describe it on the copybook.



The various proposals are written on the blackboard. Most of the descriptions highlight the alternation of the couple and of the four. Two examples:
(a) 2 black, 4 grey, we need two stamps: 

(b) We need two stamps: 

«We only need one stamp!» (also the authors of the first two stamps agree). The pupil comes to the blackboard and draws:
(c) 
✓ «Which of these solutions helps you more if you have to find the colour of the 36th circle?»
he chooses the solution (c) and writes on the blackboard: 2 + 4 + 2 + 4 + 2 + 4 and then, after a pause, specifies «And I stop»

(Continues in the following page)

Suitable age related activities	6	7	**8**	**9**	**10**	**11**	12	13	*Comments*

The author of (b), decisive element also in other occasions along with two other girls, chooses solution (c) and proposes the calculation '6 multiplied by 6' and says that the 36th circle is grey.
✓ «And the 38th?»
«6 multiplied by 6 plus 2. The second black one»
✓ «And the 92nd?»
The class shows enthusiasm towards the difficulty of calculations, but many pupils cannot solve them. We make them realise that they are working with multiples of 6. The activity is frantic. Shortly after they reach the solution:
«6 multiplied by 15 plus 2. It is the second violet one»
✓ With the collaboration of some pupils, the co-ordinator sums up that in order to find the structure of the frame, they must concentrate on the alternation between different circles, but also on the repetition of the stamp.
We propose another frame and we ask the colour of the 59th circle:

● ● ● ○ ○ ○ ○ ● ● ● ○ ○ ○ ○

A little girl proposes a misleading solution:
«We make a stamp with three black circles and two grey ones and then we turn it [2]»
We change the stamp always putting odd numbers of circles; we ask again the colour of the 59th circle.

● ● ● ○ ○ ○ ○ ○ ● ● ● ○ ○ ○ ○ ○

After some mistakes due to superficiality in the oral description of the decoration, we get to the answer
«7 × 8 + 3. The third black one»
We cannot define a formula with precision because the meeting is over and the class is in high spirits and doesn't want to stop the activity [3]. As a homework we propose a game in which we imagine that Brioshi sends a problem on the same frame and asks us what symbol is in the 82nd place.

Note 2

The following diaries show how, by favouring the identification of regularities, the stencil allows to single out the symbol associated to a given position; (for example, the drawing in 74th position). We consider it useful for the reader to explain how this search is carried out.
We consider the following decoration, whose pattern is formed by 8 elements: 5 stars and 3 moons.

★★★★★☾☾☾★★★★★☾☾☾

It is clear that the 7th symbol is a moon; the 11th is a star. In order to find the 26th symbol we must divide 26 by the number of elements in the pattern, that is 7:

26 : 7 = 3 with remainder 5

The operation is to be interpreted in this way: 3 represents the number of patterns and 5 the position of the symbol within the pattern. To conclude: the symbol is in the 5th place of the fourth pattern and thus is a star.

(Continues in the following page)

[2] The proposal is not wrong from an operational point of view, but it is not productive from the point of view of the identification of regularities.

[3] One difficulty might derive from the fact that the expression (as it has been proposed) is correct, but would be more adherent to the structure of the decoration if 7 and 8 were inverted: 8 × 7 + 3. This would make its interpretation easier.

Suitable age related activities	6	7	**8**	**9**	**10**	**11**	12	13	*Comments*

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A second example, with a much bigger number, as pupils like it, when they have understood the mechanism: what is the 1999th symbol?
1999 : 8 = 249 with remainder 7
Therefore it is the 7th symbol of the 250th pattern, that is a moon.
In fact we point out that, what really matters *is not the quotient but the remainder*, and this is not a trivial discovery for the pupils.
The following step is the identification of ‘which’ star and ‘which’ moon. We discover that in the first case it is the 5th star, in the second case, the 2nd moon.

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Diary 3 (fourth year, primary school, 9 year olds, October)

We propose a game in which Brioshi sends this frame.

★★★★★☾☾☾★★★★★☾☾☾

√ «What drawing do we find in the 82nd position? »
The class identifies the pattern, which is formed - in this case- by 8 drawings.
«82 is 8 multiplied by 10 plus 2 and it is the second star [4]»

2. Repetition of words

An activity, which is similar to the preceding ones, is based on letters, instead of symbols. If the ‘stamp’ is a word with a complete sense the exploration is maybe easier than with decorations, because the immediately perceptible meaning helps the identification of regularities. If we propose repetitions of groups of letters with no meaning, then the difficulties are similar to those we met working with decorations.

Diary 4 (fifth year, primary school, 9 year olds, December)

In his pencil-case a pupil has an advertising stamp formed by a rubber string turning round an inked pin, similar to a tracked tank which, dragged on paper, endlessly prints the writing FACCIADIMERLUZZO (litterally ‘cod-face’, a slight and funny offence generally used by very young children):

FACCIADIMERLUZZOFACCIADIMERLUZZOFACCIADIMERLUZ

√ We rise to the occasion to give this problem «What letter will be in the 244th position?»
«Facciadimerluzzo is formed by 16 letters, if I divide 244 : 16 I obtain 15 and a remainder of 4»
«Then we must look at the fourth letter, that is the second C»

[4] *As we'll see also in next diary, when the number indicating the position is small, pupils spontaneously do mental calculations and think in terms of ‘multiplication’. We must make the pupils (see Diary 2) understand that the process is always* ***multiplicative****, but that in fact it is more convenient to think in terms of division. We might help ourselves in this sense working with bigger numbers, which restrains mental calculation and favours the use of multiplication*

Suitable age related activities	6	7	8	**9**	**10**	**11**	12	13	Comments

Diary 5 (fifth year, primary school, 10 year olds, December)

The pupils are enjoying themselves looking for regularities in groups of letters. We propose the analysis of their names repeated:

STEFANIASTEFANIASTEFANIASTEFANIASTEFANIASTEFANIA

√ «What letter will be at the 150^{th} place?»
The calculation of the letters is feverish and the answers overlap.
«They are eight! 150 divided by 8!»
«And then we look at the remainder!»

Second phase: the necklace 2 grey pearls / 5 black pearls

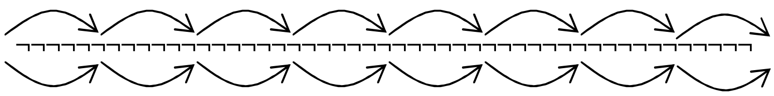
3. We start with the real necklace. Once the class has familiarised with the situation, we can pass on to its **representation** on the blackboard [5].

[5] The teacher will evaluate the most suitable moment for this passage.

Diary 6 (fourth year, primary school, 9 year olds, November)

√ We ask to describe the necklace; in front of many overlapping proposals we ask to write the description on the copybook.

We write the descriptions on the blackboard.
(a) It starts with 2 grey ones and ends with 5 black ones.
√ We ask the pupil to specify his definition. (a) is modified:
(a) The stamp starts with 2 grey ones and finishes with 5 black ones»
(b) 2 grey ones, 5 black ones to infinity
(c) It starts with 2 grey ones, then 5 black ones and then it goes on like this
(d) The stamp starts with 2 grey ones and 5 black × infinity
(e) Some stamps of 7 pearls: 2 grey ones and 5 green ones
The discussion makes the difference between the various descriptions clear; (a), (d) and (e) highlight the 'stamps' and their repetition, whereas (b) and (c) underline the alternation of the two groups of pearls to infinity [6] Many pupils don't grasp the difference between the two points of view, maybe because the authors of (b) and (c) claim they said the same thing, but they couldn't express it clearly [7].
√ The definitions are red aloud; then we ask the colour of the 50^{th} pearl. Two strategies are translated into formulas on the blackboard:
(f) $7 \times 7 + 1$ (g) $7 \times 8 - 6$
√ «How would you explain these two formulas to a pupil of another class?» In order to stimulate the answer we resort to the bus **metaphor**. On the blackboard we draw this plan representing the path:



the stops are at each multiple of 7; in order to go to a place two stops beyond, a person can choose whether to get of at the first stop or at the following one, according to which one is the closest to the place he has to go to. The class grasps very clearly the difference between (f) and (g).
«We must look for the closest multiple and then add or subtract the right number»

[6] (b) and (c) express a rudimentary detachment from concreteness, preparing to generalisation.

[7] Pupils tend to use terms like 'die' and 'stamp' because they refer to concrete objects and to experiences made in the preparatory phase. They are important because they transmit a key concept for the identification of regularities. Anyway, language ought to be refined, and gradually children should start employing the term 'pattern'

Suitable age related activities	*6*	*7*	*8*	***9***	***10***	***11***	*12*	*13*	*Comments*

Expansion 1

The teacher might take advantage from the situation of the bus stops described in Diary 6**,** to underline the property of *Archimedeicity*. This property turns out to be fundamental for divisions: 'Given two natural - and not null- numbers, there always exists a multiple of one that exceeds the other'. This is equivalent to the statement that *every natural and not null number is always included between two successive multiples of a given number*.
On an experimental basis, it is sufficient to consider how many times a number is included in the second. For example, let's consider some couples of numbers:

(14; 27) (10; 37) (5; 78) (25; 104)

In (14; 27):
$14 < 27$ but $2 \times 14 > 27$;

In (10; 37):
$10 < 37$, $2 \times 10 < 37$, $3 \times 10 < 37$ but $4 \times 10 > 37$;

In (5; 78):
$5 < 78$, $2 \times 5 < 78$, $3 \times 5 < 78$, ..., $15 \times 5 < 78$ but $16 \times 5 > 78$;

In (25; 104):
$25 < 104$; $2 \times 25 < 104$; $3 \times 25 < 104$; $4 \times 25 < 104$ but $5 \times 25 > 104$.

Such property is called *Archimedeicity*, basically because it is the one referring to magnitudes, formulated by Archimede, commonly known for *lengths of segments* (given two segments there always exists a multiple of one exceeding the other).
The habit of paying attention to the facts on which concepts and procedures are based (in this case, division) since the first years of primary school, is essential to direct pupils towards a theoretical dimension of mathematics.
Let's now compare the two diaries, one of a fourth year and the other of a fifth year that we will comment later on.
The classes are working with the initial necklace:

Diary 7 (fourth year, primary school, 9 year olds, November)

√ «What colour is the 35th pearl?»
«7 multiplied by 5 gives 35… the 35th pearl is black»
√ «And the 37th?»
Comes to the blackboard and writes, speaking aloud

$$7 \times 5 + 2$$

«The second grey one»
√ «And the 92nd?»
We make them realise they are working with the multiples of 7. After a while, here come the written solution and the answer.
«7 multiplied by 13 plus 1. It's the first grey one»

Suitable age related activities	*6*	*7*	*8*	***9***	***10***	***11***	*12*	*13*	*Comments*

Expansion 2

About the writings 7 × 5 + 2, it is important that pupils see division as a binary operation, that is an operation acting on *a couple* of numbers (in our case dividend and divisor) having as a result *another couple*: quotient and remainder.
Nevertheless, it is important to lead the pupils to express the link among dividend, divisor, quotient and remainder in several ways. For example, the division between 15 and 6 gives 2 as a quotient and 3 as a remainder. The relationships among the four number can be represented in several ways:

- $15 = 2 \times 6 + 3$
- $15 - 2 \times 6 = 3$
- $(15 - 3) : 6 = 2$

With older pupils we can aim at a generalisation: given two natural numbers a and b, with b > 0, and named q and r quotient and remainder respectively, we can express the relationships among them in several ways:

- $a = q \times b + r$
- $a - q \times b = r$
- $(a - r) : b = q$.

This transcription exercise, if applied regularly since the first approach to division, might avoid known difficulties in the formal codification of such link and favours flexibility in realising algebraic transformations.

Diary 8 (fifth year, primary school, 10 year olds, November)

«The last pearl, the seventh, is black, and if I go on like this, I find out that the multiples of 7 correspond to black pearls»
√ «What colour will the 123rd pearl be? Why?»
«I saw the result of the division of 123 by 7. [8] It's 18 with 3 as a remainder. Number 18 is not as important as the remainder, because it means that I must look at the colour of the third pearl, which is black»
√ «And if the remainder is zero?»
«If I divide by 7 and the remainder is zero, it means that the number is a multiple of 7, and since every 7 balls the pearl is black, then the remainder equal to zero, means black pearl» [9]

It might happen that the class cannot work out the *rebuilding of the field* and that we decide to recur to the stamp; the next two diaries describe this situation (it is not the same necklace, but this is not relevant in the examined case). They are kept separate, because they refer to two different meetings with the same class; the first one clearly highlights the difficulties of this passage and the second one allows to identify the positive effects of this preparatory activity on the search for regularities.
Between the two diaries fifteen days passed, during which the class teacher didn't work on this activity.

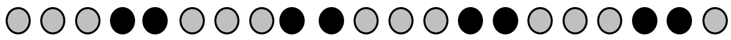
[8] We can notice that the approach to the problem changes from fourth to fifth year: in the first case pupils use the direct operation (multiplication), in the second case the inverse one (division). This passage might be interpreted as a more evolved form of thought.

[9] The activity starts many arguments not only on regularities, but also on the concept and on the meaning of division, on the theory of multiples and of divisors, on the classes remainder and, in perspective, on modular arithmetics. It is important to underline that the result of the problem is not the quotient of the division between the ordinal number of the pearl and the cardinality of the pattern, but its remainder. This is important for several reasons: because it leads to a reflection on the meaning of remainder of a division and, moreover, because it contributes to destroy the stereotype that what is on the right of the equals sign can only represent the result requested in order to solve that particular problem.

Suitable age related activities	*6*	*7*	*8*	***9***	***10***	***11***	*12*	*13*	*Comments*

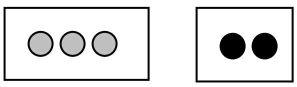
Diary 9 (fifth year, primary school, 10 year olds, November)

✓ We are working on this necklace; the class cannot identify the pattern.

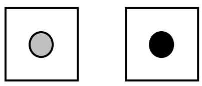


We decide to pass to a 'small frame ' and to the 'stamp'. We ask the pupils to imagine that this is not a necklace anymore but a frame and we ask them to describe what stamp they would use to draw it.
«I would make a stamp formed by two grey circles, then one with a grey circle, then one with a black circle and another one with a black circle [10]»
Another pupil goes back to the necklace and tries new calculations: «3 plus 2 makes 5, then 5 plus 3 makes 8, and so on»
✓ We ask to remain on the frame and that every pupil plans a stamp.
We obtain the following drawings, then written on the blackboard and discussed.

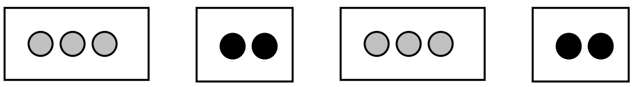
(a)



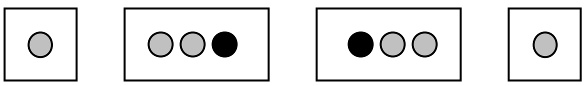
(b)



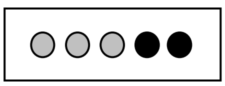
(c)



(d)



(e)

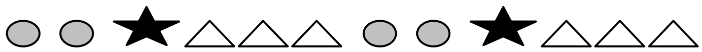


During the collective discussion the following proposals emerge:
the author of (b) declares that he thought he might use the two stamps, dip them in two different colours and then use them according to the original model to make the frame.
(c) is considered wrong because it is useless to have double the stamps.
(d) doesn't allow to obtain the original frame.
(e) is considered the correct stamp by everyone.

*[10] This pupil's apparently complicated proposal - just like those regarding some of the stamps produced later on - highlights a general aspect that further research will have to look into (see Glossary: **didactical mediator** and metaphor).*

Suitable age related activities	6	7	8	**9**	**10**	**11**	12	13	*Comments*

Diary 10 (fifth year, primary school, 10 year olds, December)



✓ We ask what the 63rd symbol is. Many pupils still tend to concentrate on colour and form. We make it clear that in this moment it is not important to find the answer, but rather to be able to explain how to arrange an answer.

A little girl understands that it is convenient to identify a stamp formed by six drawings: two circles, a star, three triangles. She comes and writes it on the blackboard.

The idea of a stamp works; the pupils talk about «repeating the stamp»; the aspect of concreteness of the gesture (repetition) accompanied by the movement of the hand.

√: «When you say you want to repeat the stamp, do you understand what you do from a mathematical point of view?» In front the class's perplexity we ask if they have ever heard of 'multiples'. The answer is yes.

Eventually, a little girl's face lightens «They are multiples of 6!»

«The stamp is formed by 6 symbols, then I repeat it and count the symbols»

√: «Good! And would you be able to tell me what is the 20th drawing?»

Some pupils continue to count a symbol after the other.

«I count 6 by 6 as far as I can and then I add …»

A small group elaborates this idea «I count 6, 12, 18 and add 2 which makes 20»

√: «Could you write down your idea for Brioshi?»

«6 × 3 + 2. Well, then it's a circle! The second drawing is a circle!»

√ «And the 57th?»

«6 × 9 + 3: it is the third drawing, it's a star!»

√ We ask them to concentrate on the search for the number of times the stamp is used. We argue all together on the fact that, using the times tables, it is simple to find the drawing if its position is given by a small number. Things get more complicated if the pearl is in a position identified by a big number. How can we sort this out?

«For example 1548. I subtract 2 and obtain 1546 which is a multiple of 6» «How do you know that such a big number is a multiple of 6?»

The pupil cannot answer. The class doesn't know how to proceed.

√ Once again, we propose a small number: the 13th drawing.

«6 × 2 + 1»

√ We try to suggest a division «What do you look at? How many times… »

«The stamp is formed by drawings. I need to know how many times I have to repeat it»

√ «And so? What operation do you have to make?» Perplexities.

√ We ask the pupils to concentrate on the search for the number of times they use the stamp. After a while, we ask what the 3576th drawing is.

Finally «We make a division!»

«I thought that if it has to be contained in it, I make 3576 : 6»

The classmates immediately understand that this is the right solution. The division 3576 : 6 = 596 is made and they find out that it is a triangle. The discussion leads to the conclusion that the core of the matter is not in the quotient, but in the remainder of the division, which represents the position of that specific drawing within the stamp.

Suitable age related activities	6	7	8	9	10	11	12	13	Comments

Third phase

Note 3

The pupils will have to face bigger difficulties than those in the preceding phase; from the logical mathematical point of view, the difficulties derive from an obstacle connected to a new interpretation of the structure of the necklace: a new *rebuilding of the field* is necessary (see **Note 1**). We stop and think about this delicate subject, so that the teacher understands the nature of this obstacle, the ways to get over it. Moreover, in this way, it will become easier to follow, through the inevitable conciseness of the diaries, the evolutions of the ideas that the classes can express.

To sum up the activity of the first phase:

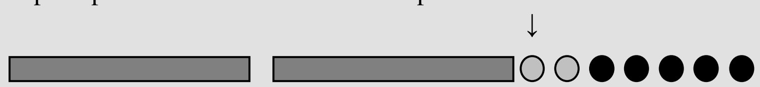
For example the question is:

(a) «What's the colour of the 15th pearl?»

It contains a datum (the 15th position) and an unknown (the colour).
The answer to the question (a) derives from the conclusion of this process:

(answer a) 15 : 7 = 2 with remainder 1

The pearl is the 1st one of the 3rd pattern and it is grey.
The perception of the necklace can be represented like this:



In simple words: in fact, the pattern is perceived as an *opaque* element; it doesn't matter what elements compose it. What matters is to realise that it is repeated a certain number of times. It doesn't matter *how many* times, because the substance of the matter doesn't change. In fact, we saw that what matters is not the quotient anymore (2) but *the remainder (1),* which tells us the pearl we look for is the first among the 'visible' ones, after the last repeated pattern.
As we underlined in Note 1 we can get to this conclusion through a *rebuilding of the field* allowing this passage:

initial perception of the necklace as *alternations of groups of pearls*
↓
perception of the necklace as *repetition of a pattern*

In the Second phase we'll ask a new kind of question:

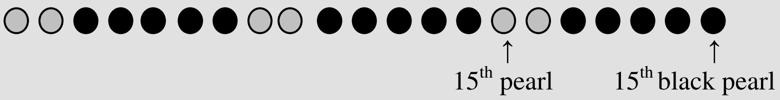
(b) «In what position is the 15th *black* pearl*?*»

In it, apart from the colour, we refer to *two* positions, one is known (the 15th pearl) and the other is unknown («In what position…»). Let's concentrate on the difference between these two positions.
It is better to represent them:

(Continues in the following page)

Suitable age related activities	6	7	8	**9**	**10**	**11**	12	13	Comments

→



In order to find the position of the pearl in the necklace we refer to the *pattern* (answer (a)) but in this way we find the nth pearl, not the nth pearl *of that particular colour*.
Now – and this makes the *new* rebuilding of the field necessary - the patterns cannot be 'opaque' anymore, but again they have to be seen as *sets of pearls of two different colours*. The passages can be schematised:

perception of the necklace as a repetition of an 'opaque' pattern
↓
perception of the necklace as a *repetition of a 'transparent' pattern*
formed by 2 *sub-patterns* of 2 grey pearls 5 black pearls respectively
↓
perception of the *repetition of the sub-pattern*
formed by the *5 black* pearls

We represent this last perception with the sub-pattern of the black pearls *in evidence*:

The 15th black pearl is then *in the third sub-pattern* and the search for it is represented by the result of this process:

(answer b) 15 : 5 = 3 with remainder 0

The pearl we look for is the 5th black one of the third pattern.
We propose another question, which does not allow a visible comparison in the drawing of the necklace:

(c) «In what position of the necklace is the 89th *black* pearl?»

The answer is:

(answer c) 89 : 5 = 17 with remainder 4

the pearl we look for is then the 4th one of the *18th sub-pattern*.
But this answer is not enough yet. In order to find the position of a pearl of a certain colour in the necklace, it is necessary to understand that, in fact, *the sub-pattern corresponds to the pattern*. Therefore, we don't need to think about the 5 black pearls of the *sub-pattern,* but take into consideration the 7 pearls of the *pattern*.
The complete answer, which allows the identification of the position in the necklace of the 89th black pearl will be found in this way:

(answer c, first part) 89 : 5 = 17 with remainder 4
(answer c, second part) 7 × 17 + 4 = 123

The 89th black pearl is in the 123rd position.

Suitable age related activities	6	7	8	9	10	11	12	13	Comments

4. We continue the activity with a necklace formed by 2 grey pearls / 5 black pearls.

Diary 11 (fourth year, primary school, 10 year olds, November)

We highlight a very frequent mistake made by pupils at the beginning of the Second phase: pupils continually apply the formula to find a certain pearl and they find out it doesn't work anymore.

The pupils are working on a necklace with pattern 7 for the first time.
√ «Pay attention. Now, I'll give a more difficult problem *[euphoria]*. In what position is the 22^{nd} black pearl?»
The answers are immediate, but feverish.
«5 multiplied by 3 minus 1»
«7 multiplied by 2 plus 1!»
«No! 7 multiplied by 3 plus 1! It's grey!»
A check of the last formula makes it clear that pupils didn't look for the 22^{nd} *black* pearl, but simply for the 22^{nd} pearl.
«7 multiplied by 4 minus 6!»
«It's the 22^{nd} pearl again»
Pupils count the pearls drawn on the blackboard and they find out that the 22^{nd} black pearl is in the 29^{th} position. In the end, they get to the correct formulas, which we immediately write on the blackboard.

$6 \times 5 - 1 \qquad 6 \times 4 + 5$ [11]

Diary 12 (fifth year, primary school, 10 year olds, December)
Generally, the search for something stimulates the pupils' creativity very much. They elaborate a great number of strategies at various levels of efficacy. Comparison and discussion favour the identification of the best strategies.

√: «In what position is the 10^{th} grey pearl?»
«In the 30^{th} position: I counted in my drawing [12]»
« I thought about the times table of 2, and I saw that it is in the 5^{th} group [13]»
«I looked at the blackboard *[22 pearls are drawn, the last one is the 7^{th} grey one]*. If the 7^{th} grey pearl is in the 22^{nd} position, the 8^{th} grey one is in the 23^{rd} position. I'm short of 2 for the 10^{th}, I add 7 and find the 30^{th} position [14]»
«There are 4 groups of 7 before the 10^{th} pearl: 7 plus 7 *(he thinks)* plus 7 plus 7. Yes, it's true, it is in the 30^{th} position [15]»
√: «Well, then let's find out in what position is the 100^{th} grey pearl»
«The 100^{th} grey pearl is the second one of the 50^{th} pair. That means that before it, there are 49 pairs. But then I have to add 2 [16]»
«If I look for the black pearls, I multiply 49 by 5 which gives 245. Then, before the 100^{th} grey pearl there are 245 black pearls. I add the 100 grey pearls, and obtain 345 pearls. the 50^{th} grey pearl is in the 345^{th} position [17]»

[11] It is evident that pupils managed to build the two expressions because they drew it on the blackboard. They are still far from the solution, but it is important to stir things up. In the following meetings we'll go back on the unfinished formulas and we'll find out their generalisations.

[12] It is a common strategy. If the number is not too big there are always some pupils who - with painstaking patience - draw even 100 or 200 pearls and then count them (also see Diary 12).

[13] The pupil identifies the sub-pattern of grey pearls and on this base, he elaborates his argument. It's worth noticing that the strategy is once again of multiplicative kind (the times table of 2) and the division is not perceived.

[14] The strategy is heuristic; it starts by counting directly on the drawing and then it goes on with successive additions until the conclusion.

[15] The pupil does a test of the argument in discussion through an addictive process.

[16] Also this one is a reflection made aloud on the repetition of the sub-pattern formed by the pairs of black pearls.

[17] The pupils elaborate their arguments on the basis of the two sub-patterns, which - if summed up- give the position we look for.

Suitable age related activities	6	7	8	**9**	**10**	**11**	12	13	*Comments*

Diary 13 (fifth year, primary school, 10 year olds, November)
In order to make the initiation to exploration easier, we start by asking questions on visible pearls, and then we pass on to other non-visible pearls. In this case, the class reaches a more advanced conclusion then the preceding ones.

√ «In what position is the 3rd grey pearl?»

The answer is immediate «It's the 8th pearl, I counted and I saw that the third grey pearl is in the 8th position!»

√ «In what position is the 10th grey pearl?»

The pupils think for a longer time because they have to 'add' the non-visible pearls.

«I counted the 10 grey pearls, and then I added the black ones that are in between: 4 groups of 5, and so the 10th grey pearl is in the 30th position»

«Yes, so did I: 2 multiplied by 5, I multiplied by 4 and I got 28. Then I added another 2 grey pearls, the 9th and the 10th one, I get to the 30th position»

√ « And the 31st grey pearl, where is it?»

«If the 10th grey pearl is in the 30th position, the 20th grey pearl is in the 60th position, the 30th grey pearl is in the 90th position, and so the 31st grey pearl is in the 91st position»

It seems to work, but some pupils do not agree.

«No, because the 31st grey pearl is not immediately after the 30th, but there are 5 black pearls in between.» [18]

«We can't go from 10 to 10, we don't have patterns of 10!»

Some minutes pass and pupils work individually. Then, little by little they start speaking and compare their ideas.

«The pearl is in the 106th position: I drew and counted them one by one»

Her classmates glance at this girl's copybook.

A pupil makes an observation, which opens up the situation.

«Why can't we count by 7, since we have patterns of 7?»

«With 31 grey pearls I build 15 pairs of grey pearls, and the 31st is the one coming right after them»

«It's true, I did 15 by 7… 105, and so the 31st grey pearl is the 106th in the necklace [19]»

√ «So, in what position will the 43rd black pearl be?»

«The black pearls are in groups of 5. With 43 black pearls I can have 8 groups of 5 pearls and 3 are left, so it is the 3rd pearl after 8 complete groups [20]»

The answers of a group of students enhance the organisation of a last proposal in a complete form, on the blackboard:

$5 \times 8 = 40$	black pearls in 8 groups
$2 \times 8 = 16$	grey pearls in 8 groups

And they conclude «Over all, 8 groups are 56 pearls, plus two grey pearls, they are 58. The third pearl coming after them is the 61st [21]»

[18] *At this stage some pupils elaborate a very complex strategy, of a proportional kind. A later intervention proves it wrong, showing its incorrectness. This is a significant example of how a discussion in class contributes to conceptualising and favours a social building of knowledge.*

[19] *The pupil grasped the passage from sub-pattern to pattern (see **Note 3**).*

[20] *Also this pupil realises the passage sub-pattern / pattern.*

[21] *In the two expressions written on the blackboard a perception of the two alternating sub-patterns prevails. In the conclusion the passage from a repetition of the sub-patterns to the repetition of the pattern is realised.*
*The expression $7 \times 8 + 3 = 61$ is implicit (see **Note 4**).*

Note 4

The comparison between representations

$5 \times 8 = 40 \qquad 2 \times 8 = 16 \qquad 7 \times 8 = 56$

offers a hint to face or study in depth the distributive property, in a context in which its meaning is visible. We can ask the pupils to reflect on the equivalence deriving from the same ‘effect’ of the two paths followed in order to identify the pearl:

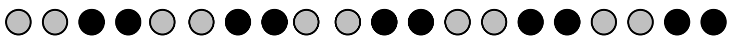
$$5 \times 8 + 2 \times 8 = 7 \times 8$$

and argue in terms of a further change of the representation:

$$5 \times 8 + 2 \times 8 = 7 \times 8 = (5 + 2) \times 8$$

Fourth phase
The necklace 2 grey pearls / 2 black pearls

5. We pass on to a necklace in which pairs of grey pearls are alternated to pairs of black pearls. A higher degree of regularity makes it more difficult to study, despite the initial appearance, because the attention is drawn on the alternation of the two *sub-patterns*. The field (in a *Gestalt* meaning) is, so to speak, perceptively *neuter* and it slows down a rebuilding. For these reasons, it represents a positive occasion for a test and a reflection on the achievements realised so far by the class.



6. The first reflection is about the position of a pearl within the pair. For example:
- Is the 7th pearl the first or the second of the pair?
- And the 27th?
- And the 102nd?

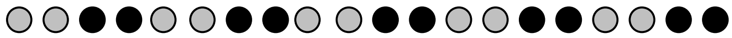
Diary 14 (fifth year, primary school, 10 year olds, November)

√ «Is the 24th pearl the first or the second of the pair?»
The class thinks.
«It's the second»
√ «And is the 9th the first or the second of a pair?»
«It' s the first»
√ «How did you understand it?»
«If the number is odd, the pearl is the first of the pair, if it's even, it's the second»

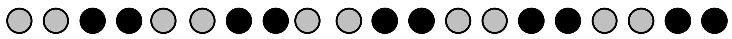
Suitable age related activities	6	7	8	9	10	11	12	13	Comments

7. We continue by drawing the attention on the relationship between the position of the pearl and its colour, as a consequence of the alternation of the colours (for example: the 2nd pearl is grey, the 4th is black, the 6th is grey and so on).

Diary 15 (fifth year, primary school, 10 year olds, November)



«How odd, since the10th pearl is grey, I thought the 20th would be grey, but it's black. In the first tenth you start from the grey one, but then, in the second tenth, it's the same as if I started from the black one»
We draw this on the blackboard:

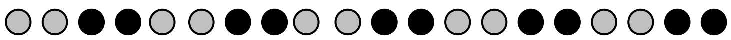


«The 20th pearl is black, I can see it. So, colours are alternated for each tenth»
The discussion leads to more and more articulated discoveries:
«Yes, the even tenth is black, the odd tenth is grey»
«We must look at the figure of tenths»
«If the number of the pearl ends in zero, in order to identify the colour, it is necessary to study the figure of tenths: odd grey, even black»
«The pearls corresponding to a number ending in two or more zeros are always black»

7. Our work goes on looking for the colour of the pearls according to the position within the necklace:

- What colour is the 10th pearl?
- What colour is the 35th pearl?
- What colour is the 123rd pearl?

Diary 16 (fifth year primary school [22], 10 year olds, December)

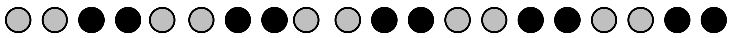


«The 10th pearl is grey, because I tried to count, the 10th turned out to be grey»
«I started with two grey ones. I counted how many groups of two, how many pairs there must be to get to 10. I found 5 because 10 divided by 2 is 5. Then I counted: grey-black-grey-black-grey: the 5th pair is grey, the 10th pearl is grey»
«In my opinion the 10th is grey because 10 is even»
«No, it's not for this, because there are even pearls that are black, just look at them»

[22] *It is a different fifth year class from that of diary 15*

Suitable age related activities	*6*	*7*	*8*	***9***	***10***	***11***	*12*	*13*	*Comments*

Diary 17 (fourth year, primary school, 9 year olds, December)
As in Phase 1, pupils need to identify the pattern of 4 pearls: This passage is necessary for generalisation (it is always the same necklace).



«We might try the times table of 4; with 4 we always find the black one»
We test that the hypothesis works.
√: «Can you find out the colour of the 25th pearl?»
A child comes to the blackboard and proceeds step by step. He writes:

$$16 + 4 = 20$$

He realises this is not enough and writes underneath:

$$20 + 4 = 24$$

He realises it is still not enough and writes:

$$24 + 1 = 25$$

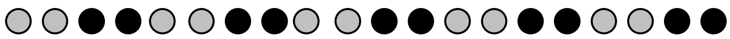
He finally finds out that the pearl is grey.
√ «How could we describe the rule?»
«In this necklace every multiple of 4 is the second pearl of the black pair»
√ «Is there a way to describe in mathematical language the colour of the 25th pearl for Brioshi, by using the rule expressed by Rossella?»
They reach this collective writing on the blackboard:

$$4 \times 6 + 1$$

√ «And how do you find the 38th pearl?»
The class expresses self-confidence and writes:

$$4 \times 9 + 2$$

Diary 18 (fifth year, primary school, 10 year olds, January)
We think it is interesting to propose the strategy elaborated by a girl in a fifth year.



√ «What colour is the 32nd pearl? Why?»
she uses her drawing and explains her strategy:
«The first two grey pearls represent the two ends of the first tenth. It starts with a grey one and ends with another grey one, the second pair of pearls represents the second tenth, it starts with black and ends with black, the third pair is the third tenth, then I advance of other two pearls and find a black pearl. This means that the 32nd pearl is black» [23]

[23] *While reading her drawing, the little girl rebuilds the field in an absolutely original way. She realises that she can opacify the pearls lying between the ends of every tenth*

and she builds a reduced model of a tenth composed by a pair of pearls whose colour is the same as that of the ends of the tenth it represents. She counts the pairs until an approximation by defect to less than 10 the number she's looking for; then she reassigns the pearl its original role of unit. We think this is a good example of how an activity rich in hints stimulates at the same time creativity, rationality and language

Suitable age related activities	*6*	*7*	*8*	***9***	***10***	***11***	*12*	*13*	*Comments*

Fifth phase

8. The activity is the same as that proposed in the Third phase. We try to connect grey or black pearls to the position occupied within the necklace. We'll ask questions such as:

- In what position is the 3rd black pearl?
- And the 3rd grey one?

Now pupils are ready to pass to a generalisation.

Diary 19 (fifth year, primary school, 10 year olds, February

○○●●○○●●○○●●○○●●○○●●

√ «In what position is the 3rd black pearl?»

The answer to this question is simple because we can easily see that the pearl is in the 7th position.

√ «In what position is the 9th black pearl?»

It is in the 19th position.

A pupil proposes this calculation $9 + 9 + 1$, in order to find it

√ «What colour is the15th black pearl?»

This time it is more difficult because the 15th black pearl is not drawn on the blackboard.

√ We propose to start an ordered search by filling in a table (it is an instrument that pupils know). In the first column we'll write the ordinal number of the black pearl and in the second its relative position in the necklace [24].)

number of the black pearl	position in the necklace
1	3
3	7
5	11
7	15
9	19

√ We invite the pupils to find how we can pass from the numbers on the left to the numbers on the right. Shortly, the 'rule' is discovered and written on the blackboard.

Table 1

number of the black pearl	position in the necklace	'rule'
1	3	$1 \times 2 + 1$
3	7	$3 \times 2 + 1$
5	11	$5 \times 2 + 1$
7	15	$7 \times 2 + 1$
9	19	$9 \times 2 + 1$

The conclusion is common: the 15th black pearl is in the position

$$15 \times 2 + 1 = 31$$

√ «Can you find out in what position is the 30th black pearl?»

«In the 61st position!»

The collective test brings to the conclusion that it is not true: the 30th black pearl is not in the 61st but in the 60th position. The formula pupils found is only valid for black pearls that are in a position represented by an odd number.

Then let's build another table:

(Continues in the following page

[24] In this table, we decide to insert the pearls occupying odd positions in order to facilitate -as will result clearly later on- the pupils' search.

Suitable age related activities	*6*	*7*	*8*	***9***	***10***	***11***	*12*	*13*	*Comments*

→

Table 2

Number of the black pearl	position in the necklace	'rule'
2	4	2×2
4	8	4×2
6	12	6×2
8	16	8×2 [25]

√ At this stage we propose to describe in simple words some *General Rules* related to the black pearls.

«If the number of the black pearl is odd, its position is the double plus 1» *[Table 1]*

«If the number of the black pearl is even, its position is the double» *[Table 2]*

√ We ask the pupils to test with a table if it is valid also for grey pearls.

Table 3

Number of the grey pearl	position in the necklace	'rule'
1	1	$1 \times 2 - 1$
3	5	$3 \times 2 - 1$
5	9	$5 \times 2 - 1$
7	13	$7 \times 2 - 1$
9	17	$9 \times 2 - 1$

When established that the rule changes, we complete the table of grey pearls in even position.

Table 4

Number of the grey pearl	position in the necklace	'rule'
2	2	$2 \times 2 - 2$
4	6	$4 \times 2 - 2$
6	10	$6 \times 2 - 2$
8	14	$8 \times 2 - 2$
10	18	$10 \times 2 - 2$

Given the table, the conclusions are:

«If the number of the grey pearl is odd, its position is the double minus 1»

«If the number of the grey pearl is even, its position is the double minus 2»

√ We ask to translate the Rules into mathematical language for Brioshi, so that according to the colour, it is possible to find the position of any pearl in the necklace.

Pupils choose letter 'p' to describe the position of the pearl [26].

black pearl	table 1	odd number:	$p \times 2 + 1$
	table 2	even number:	$p \times 2$
grey pearl	table 1	odd number:	$p \times 2 - 1$
		even number:	$p \times 2 - 2$

[25] *The reason for which we preferred to keep the pearls in even positions separate from those in odd positions, is now clear: this made regularities much easier to be identified, because there is no interference*

[26] *The class has already met letters in maths, because it's the second year it takes part in the ArAl project*