Unit 6

From the scales to the equations

5rd – 6th grade

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ArAl Project	U6. From the scales to the equations

1. The Unit
The approach to equations described in this unit takes a starting point from some research described (among others) in: Da Rocha Falcâo J.T. (1996) and in Dias Schliemann & al. (1993). Our research results on this subject are explained in Navarra (1998), Navarra (2001), Malara e Navarra (2001).

2. Didactic aspects

The activity points to an approach to algebraic thought through the initial use of scales. Some processes of collective building of knowledge are activated; the pupils elaborate and compare different **representations**, they improve abilities regarding **translation** from natural into symbolic language and vice versa, they make operations properties explicit, they get used to **letters** as **unknowns**.

3. General aspects

In order to favour the passage from arithmetic to algebra, many studies have explored the use of culturally significant **metaphors** like the scales.
In Italy scales are proposed in some maths textbooks of Junior High school, mostly in the exercise section, as a support to the activities on equations. Anyway, its use is only occasional, and the concreteness it evokes is merely *virtual,* meaning that it wants to support the passage to abstraction with a *realistic* device. Nonetheless this device remains only a representation of the object, very often not even a familiar one for the pupil. Scales should imply an *experimental* attitude, but in this way it isn't really *experimented.* The outcome is probably that it is used only in order to propose a different problem, but it doesn't allow the pupil to interiorise its conceptual aspects, and so it misses the target for which it was adopted.
Research in the field of maths didactics has extensively treated scales, and it has underlined the risks deriving from a prolonged use, mainly connected to the possible creation of stereotypes for the students, and hence some conceptual fixity, or worse, a wrong concept. This might represent a hindrance or a distortion to the coherent development of algebraic thought.
From the results so far collected, our aim appears to be plausibly reachable: verifying how in the solution of verbal algebraic problems, a specific use of scales, integrated with a suitable use of representation, can be a positive approach to the development of pre-algebraic patterns in the pupils. It can also allow the elaboration of **'hybrid' equations** in which – in a temporary initial phase – a coexistence of *natural language, iconic language and formal mathematical operators* is possible.
Through the solution of suitable sequences of problems with scales first, and then verbal problems, the itinerary is organised on the **collective** building of the concept of the linear equation, seen as the final point of a **process**, centred on subsequent schematisations of representations of situations initially proposed with scales.
Systematic attention is paid to the *variety* of representation possible for a given object; in more and more advanced ways, in the passage from elementary to junior high school, the pupils are led to understand how *the choice of a representation influences the development of argumentations on the represented object* and how the effectiveness of each is strongly related to the environment where the object is, and to the aims of the observer.

4. Necessary materials

Scales; a kit of unknown and known weights; in the examples reported in the unit, the weights are boxes labelled with words such as 'flour', 'salt', 'rice', and 60', '120', '150', '200', '270'. The precision in the construction of weights is not necessary because, almost immediately the scale pans are blocked in the position of equilibrium (see**. Comment 1**, **situation 1**) and so incidental imperfections are not detected.

5. Terminology and symbols

Phase	Sequence of situations of growing difficulty referred to the same subject.
Situation	Problem around which individual, group or class activities are developed.
Expansion	Hypothesis of work on a possible expansion of the activity towards an algebraic direction. Its realisation depends on the environmental conditions and on the teacher's objectives.
Supplementary activity	Enlargement towards subjects related to those developed in the preceding **Situations**.
Note	Methodological or operational suggestions for the teacher.
	In the square a problematic situation is proposed. The text is purely indicative; it can also be presented as it is, but generally its formulation represents the outcome of a **<u>social mediation</u>** between the teacher and the class
<u>Representation</u>	An underlined word in boldface type highlights a link to a subject illustrated in the Glossary.

Square containing the outline of a typical discussion; the following symbols may appear:

√	Intervention of the teacher
	Intervention of a pupil
	Summary of several interventions
	Summary of a collective discussion (a principle, a rule, a conclusion, an observation, …)

6. Phases, situations and subjects

All the phases are abundantly supported by significant extracts from the **<u>Diaries</u>** of activities in the classes.

PHASES	SITUATIONS	TOPICS
First	**1 - 6**	Activities with scales
Second	**7 - 12**	Representation of 'problems with scales'
Third	**13 - 17**	Verbal problems with scales as an object
Fourth	**18 - 22**	Verbal problems with iconic support, which can also be solved with an equation
Fifth	**23 - 31**	Verbal algebraic problems

7. Distribution of the situations in relation to the age of the pupils

The unit can be proposed in its entirety to the fourth and fifth year of primary school and in the first year of intermediate school, without any significant variation either from the point of view of the contents, or that of times.
Although it has not been experimented in the third years (primary school), we think that the First phase might also be performed with pupils of this class, and might thus represent a possible background for deeper studies in the following years.

			PHASES AND SITUATIONS																															
			I						**II**						**III**					**IV**					**V**								
Sch	A	C	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**	**21**	**22**	**23**	**24**	**25**	**26**	**27**	**28**	**29**	**30**	**31**
Prim.	**10**	**5**	7 hours						3 hours						3 hours					2 hours													
Inter.	**11**	**1**	5 hours						3 hours						3 hours					4 hours													
	12	**2**													3 hours					5 hours					4 hours								
	13	**3**																		4 hours					6 hours								

ArAl Project	U6. From the scales to the equations

Suitable age related activities	*6*	*7*	*8*	*9*	***10***	***11***	*12*	*13*	*Comments*

First phase

1 [1] An artisan balance is built (a fulcrum and an arm holding two equal weights at its ends) and the teacher discusses with the pupils the *conditions for the equilibrium.*

The discussion leads to the common formulation of a conclusion: if the weights are equivalent, the balance is in the position of equilibrium. If the weights are not equivalent the balance is *not* in the position of equilibrium. An important conclusion is drawn (and jotted down)

Fundamental principle of the balance:

The balance is in equilibrium if and only if the weights in the scale pans are equivalent.

Note 1

As illustrated in the ***Comments***, the beginning activities are *not* experimental, and the equilibrium of the scale pans doesn't need to be maintained. For this reason it is not necessary to prepare 'real' weights, but rather objects which 'represent' weights: little boxes or packets with labels like 'salt', 'flour', '200g', '120g', even if the labels don't correspond to 'real things'. The suggestion is: for the first two situations it is better to prepare some materials that allow the reaching of a 'visible' equilibrium. This favours the perception of horizontality between the scale pans; then, once the balance is 'blocked' in a condition of equilibrium, it will be possible to continue with virtual weights. Of course, the **didactic contract** must be fixed with the class.

Now, the activity with the scales can begin.
Every time the teacher organises the situations, he moves the 'weights', removes them, adds some more according to the indications of the pupils observing, proposing, explaining. The experiences can be repeated more than once to help comprehension. The class can intervene in the manipulation of the objects. In this phase only the conclusions drawn each time are jotted down on the copy books, and they will be called *principles*. [2]

First phase

[1] *The aim of this experience is to introduce the pupils to the idea of equilibrium as a* central *aspect of the activity: from now on* the balance is to be considered in equilibrium *whether the scale pans are lined up or not.*
The fundamental principle *will have to be frequently repeated in order to help reinforce the idea in the pupils' minds. The scholastic scales, rather unstable, are not of great help in this cases. It is better to adopt some stratagems to divert the pupils' attention from the equilibrium indicators (generally pointers). This equilibrium is very difficult to reach during the experiment and, moreover, in this activity it doesn't represent a didactic objective. For example, the teacher can agree with the class to* block the scale pans *in order to avoid their swinging. The didactic agreement must be clear: the equilibrium is a constant,* whether it is concretely reached or not.

[2] *The principles correspond to those known as 'equivalence principles' in the didactic of equations.*

Suitable age related activities	*6*	*7*	*8*	*9*	***10***	***11***	*12*	*13*	*Comments*

2. The first experience is proposed

left scale pan	*right scale pan*
salt [3]	200g

<u>**Diary**</u> **1** (fifth year, primary school, 10 year olds, October)

√ How much does the salt weights?
The salt weights 200 grams.
√ How did you find this number?
: It's evident!
√:Try to explain [4].
Most of the times the pupils don't understand what the teacher expects of them, because the situation appears evident to them. It is necessary to draw their attention towards the equilibrium.
Conclusion:
(1) as the balance is in a position of equilibrium,
(2) according to the fundamental principle, the weights are equivalent;
(3) it can be deduced that the salt bag weights 200 grams.

3. Second experience

left scale pan	*right scale pan*
flour; 50g	120g

Diary 2 (fifth year, primary school, 10 year olds, October)

√: How can we find out how much the flour weights?
: The flour weights 70 grams.
√: How did you get this number?
: *We made* 120 minus 50 and *we found* the flour [5]
√: But if you remove 50 grams only on one side, the balance is not in equilibrium anymore. We must respect the fundamental principle.
Also in this case the pupils need to be directed towards a line of reasoning that takes equilibrium into consideration [6].
We conclude that:
(1) as the balance is in equilibrium
(2) the weights are equivalent (fundamental principle);
(3) if we remove the same weight from both the scale pans, the balance remains in equilibrium and this means that
(4) the remaining weights are equivalent;
(5) hence the flour weights 70 grams.
The consideration we made explicit in point (3) is recorded in the

First Principle of the balance
If we remove equivalent weight from both the scale pans the balance remains in equilibrium [7].

[3] *For typographical reasons we indicate with the word 'salt' the box of salt, as from now on 'flour' will stand for a packet of flour and so on.*

[4] *It is important to pay attention to reflexion and to* ***<u>verbalization</u>****; so it is inevitable that the activities proceed very slowly, especially at the beginning.*
We underline again that the equilibrium must be constantly explicit, because the pupils tend solve problems intuitively *and they also tend to concentrate only on the operations.*

[5] *The slang expressions reported in the Italian version of the discussions (for example: 'We made') are very frequent orally, and they should be corrected. Also in sentences like '… we found the flour', the vagueness indicates a weak control on the underlying concepts, because what is really found is not the 'flour', but 'the weight of a packet of flour'. The passage from the* object *to its* weight *is very delicate and needs to be continually reinforced.*

[6] *It helps, at the right moment, to substitute the 120g weight with two weights of 70g and 50g. This change of the initial situation favours the comprehension of the following conclusion:*

left scale pan	*right scale pan*
flour 50g	*70g 50g*

[7] *Very often we need to enlarge the First Principle, because the pupils realise that it maintains its validity even when the same weight is added in both the scale pans.*

Suitable age related activities	*6*	*7*	*8*	*9*	***10***	***11***	*12*	*13*	*Comments*

4. Third experience

left scale pan	*right scale pan*
salt; salt	200g

Diary 3 (fifth year, primary school, 10 year olds, October)

√: How can we find out how much the single salt box weights?
: The salt weights 100 grams.
√: How did you get this number?
: We made a division.
√: If you divide only the 200 grams of the left scale pan the balance is not in equilibrium anymore. Remember the fundamental principle? And don't forget the first principle of the balance [8].
We conclude that:
(1) as the balance is in equilibrium,
(2) the weights are equivalent (fundamental principle);
(3) if we divide by 2 on the left, then we have to divide by 2 also on the right, so to maintain the balance in equilibrium;
(4) hence the salt weights 100 grams.
Then we repeat the experience, with some changes in the initial situation:

left scale pan	*right scale pan*
salt; salt; salt	300g

It is clear that in the case we must divide by 3.
The consideration in point (3) can be generalised into the

Second Principle of the balance
If we divide by a same number the contents of the two scale pans of a balance in equilibrium, it remains in equilibrium.

[8] *This situation is more complex than the preceding one. The pupils tend to conclude hastily that the container weights 100g. The explanation is once again intuitive: «We divide 200 by 2». We can show that in doing so, the* balance *is not in equilibrium anymore; referring to the first principle helps (i) to understand that it is necessary to divide (in this case by 2) the content of* both *the scale pans (ii), so to gain the Second Principle.*

Suitable age related activities	6	7	8	9	10	11	12	13	Comments

5. Fourth experience

left scale pan	*right scale pan*
150g; salt	salt; salt; salt; salt

Diary 4 (fifth year, primary school, 10 year olds, November)

√: How can we find out the weight of a packet of salt? [9]

The pupils understand that it is necessary to:
(I) apply the principle, this time to an *unknown* quantity
(II) apply the second principle to the remaining ones.
We conclude that:
(1) as the balance is in equilibrium,
(2) the weights are equivalent (fundamental principle);
(3) we apply the first principle before (we take away a bag of salt from each scale pan) and then
(4) the second principle (we divide by 3);
(5) hence the salt weights 50 grams [10].

6. Fifth experience

left scale pan	*right scale pan*
270g; rice; rice	rice; rice; rice; rice; rice; 60g [11]

Diary 5 (fifth year, primary school, 10 year olds, November)

√ How can we find out the weight of a packet of rice?
The pupils understand that it is necessary to:
(I) apply the first principle both to <u>known</u> and to <u>unknown</u> quantities, and then:
(II) apply the second principle to the remaining ones [12].
We conclude that:
(1) as the balance is in equilibrium
(2) the weights are equivalent (fundamental principle) [13]
(3) if we apply the first principle (we take away from both scale pans 2 packets of rice and 60 grams) and then
(4) we apply the second principle (dividing by 3);
(5) we find out that the rich weights 70 grams.

[9] *Frequently, at the beginning, some pupils propose to "divide 150 by 4". The* ***<u>discussion</u>*** *requests a continual* ***<u>mediation</u>*** *of the teacher who has the task of favouring an aware use of principles. It is recommendable that the class's proposals are immediately 'translated' into practical actions. This helps to verify their correctness.*

[10] *The solution of an equation involves the isolation of the unknown in a member and of the numerical values in the other. This aspect represents one of the main obstacles that pupils have to face, probably because most of them are still looking for the operations to perform. It is difficult for them to understand that, in order to answer the initial question 'How much does the packet of salt weight?', it is necessary to have* only *unknown weights on one of the scale pans and* only *known weights on the other. The experience with the balance helps a lot in the gradual comprehension of this aspect, drawing the attention on the connections among the elements of the problem (*<u>**relational**</u> *point of view) and drags the attention from the search for calculations (*<u>**procedural**</u> *point of view).*

[11] *The unknown weights are more on the second scale pan than in the first, because it is important that the pupils don't assume the stereotype that the unknown must always stay on the left.*

[12] *The discussion is often laborious; the teacher's task is often that of favouring the achievement of the solution without taking the class's place in the most delicate moments.*

[13] *At the right moment it is advisable that - on a class's indication or as a teacher's suggestion – the 270g weight is substituted by two weights of 210g and 60g (see* ***Comment*** *[2],* ***Situation 3****).*

Suitable age related activities	*6*	*7*	*8*	*9*	***10***	***11***	*12*	*13*	*Comments*

Second phase

7. We pass now on to the representation [14] of the five experiences on the copy books. We start by inviting the class to propose a *drawing of the balance and of the objects appearing on the scale pans (known and unknown weights).*
Every pupil makes his own drawing, then copied on the blackboard. The representations are various. They reflect two opposite attitudes, variously intersecting:
(a) *descriptive*
(b) *interpretative.*
The drawings of type (a) are realistic, detailed, rich and often redundant in particulars. Their main aim is 'to let you see' the objects used in the experiences clearly (the balance, the packets, the boxes and so on). The point of view is that of *concreteness*.
The drawings of type (b) are schematic, simpler than those of type (a), non redundant; the objects are represented with symbols (rectangles, squares, letters). They express an attitude towards *abstraction*.
The drawings are compared and the class starts a discussion on them, with the aim of reaching a shared representation.

Diary 6 (fifth year, primary school, 10 year olds, November)

[15] The drawings express highly mixed (and often ambiguous) points of view. It is evident that, according to many pupils, letters put instead of unknown weights don't represent numbers, but only initials of **names** ('s' for 'salt', 'f' for 'flour') or abbreviations; or they have the function of **labels**, often put into rectangular or square icons which represent the objects used in the experience (such as 'the box'). The known weights are represented both with and without the indication on the unit of measurement (this aspect will be discussed later).
Slowly the class excludes some representations, and often the choice is painful: the 'descriptive' pupils consider the drawings of group (b) poor in information and some offer strong resistance, when asked to accept as a 'value' what might be defined as a 'figurative rarefaction'. Anyway, as the drawings decrease, those, which remain result to be more synthetic although rich in significance, they are awarded for their *symbolic* value.
Nevertheless, it must be clear to everybody that the comparison ends with the choice of the representation, which is more coherent with the key concept of the activity, that is the concept of *equilibrium.*
We finally arrive at a representation like the following [16]:

=
△

Whatever drawing is chosen, it must contain *the symbol* **'='** [17].

Second phase

[14] The representation is the basic passage from the balance to the equation and it favours the approach to algebraic thought.
*When facing a problem, pupils tend to look for operations and for their results, as a habit. The aim of representation is to guide the pupils to identify the relational aspects inside the problem, delaying the search for a **result** to a successive moment.*

[15] The discussion schematised in the box is only approximate and it depends on whether or not the class has already met the use of literal symbols in mathematics.

Note 2

*If the pupils have never met the use of letters before, they might conclude that it is indifferent to use iconic or literal symbols to indicate the unknowns. This is true on one hand, but it is advisable to encourage the use of letters, more profitable for the development of **pre-algebraic thought.** The choice of a letter can be free, so the pupils are not induced to think that the unknown must* always *be indicated with the letter x.*

*[16] The scheme will be used until the pupils realise (alone or helped by the teacher) that it is useless, because the only significant symbol is the **equal sign**.*

[17] In a fifth year, before the symbol '=' was suggested, a girl proposed a vertical axis to represent the symmetry *between the two parts. She showed her intuition of a theoretical aspect, with important practical consequences: for example, it allows to* swap *the contents of the scale pans.*

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In the following situations the five experiences, summed up one by one (the balance is to be used only if necessary) will be recorded on the copy books applying the balance scheme, pointed out by the class at the end of the discussion. The things to be represented will be:
(i) the initial situation,
(ii) the actions performed, the principles applied and the consequences
(iii) the conclusion (individuation of the unknown value).

8. First experience

left scale pan	*right scale pan*
salt	200g

Diary 7 (fifth year, primary school, 10 year olds, November)

In general, we obtain a writing of this type, without any difficulty:

a = 200 **18**

9. Second experience

left scale pan	*right scale pan*
flour; 50g	120g

Diary 8 (fifth year, primary school, 10 year olds, November)

We must find a way of representing the fact that we must take away 50g from each scale pan (eg. we apply the first principle).
The biggest complication regards the right part of the writing: 120g. Very often the pupils propose the factorization of 120 in 50 + 70. As often, they propose to *cancel* in order to represent the action of *taking away* the 50-gram weight from the scale pans.

p + 50 = 120 **19**
p + ~~50~~ = ~~50~~ + 70
p = 70

This is the possible **writing** (other very common representations are in the **Comments 20**):
The following convention is commonly agreed on: in order to make the process as **transparent** as possible (after applying a principle), in the following line we must always represent what is left in the preceding line.

18 *Initially the pupils write the unit of measurement; as time goes on, they understand that it is not necessary to put into the equation; it is sufficient to indicate it in the final answer. The* ***structure*** *of the equation remains the same, no matter whether it is expressed in milligrams or tons).*

19 *Mainly in the initial phases, the symbol '+' appears in very few* ***protocols****, although sufficient to make the pupils think about the necessity of its introduction in the writings. It's an important occasion to introduce the passage from symbols as representations of objects (the letter indicating the packets, put one next to the other like the real objects on the scale pans) to symbols as nouns of numbers (weights).*

20 *Other common representations of the First Principle are:*

(a) p + 50 = 120
p + ~~50~~ – ~~50~~ = 120 - 50
p = 70

(b) p + 50 = 120
↓- 50 ↓- 50
p = 70

Suitable age related activities	*6*	*7*	*8*	*9*	***10***	***11***	*12*	*13*	*Comments*

10. Third experience

left scale pan	*right scale pan*
salt; salt	200g

Diary 9 (fifth year, primary school, 10 year olds, Genuary)

It's a very delicate moment, because it is now necessary to *represent the application of the second principle.* We note down one of the most common final representations:

$$s + s = 200$$
$$2s = 200 \quad \text{[21]}$$
$$2s : 2 = 200 : 2$$
$$s = 100$$

Diary 10 (fifth year, primary school, 10 year olds, Genuary)

11. Fourth experience

left scale pan	*right scale pan*
150g; salt	salt; salt; salt; salt

$150 + \cancel{a} = \cancel{a} + a + a + a$ [22]
$150 = a + a + a$
$150 = 3a$
$150 : 3 = 3a : 3$
$50 = a$ [23]

Diary 11 (fifth year, primary school, 10 year olds, Genuary)

12. Fifth experience

left scale pan	*right scale pan*
270g; rice; rice	rice; rice; rice; rice; rice; 60g

$270 + \cancel{b} + \cancel{b} = \cancel{b} + \cancel{b} + b + b + b + 60$ [24]
$270 = b + b + b + 60$
$\cancel{60} + 210 = b + b + b + \cancel{60}$
$210 = 3b$
$210 : 3 = 3b : 3$
$70 = b$

[21] *Many pupils feel more comfortable with an* ***additive*** *representation (s + s) than with a* ***multiplicative*** *one (2s, s x 2, 2 x s, …) This last one implies conceptual and subtle difficulties, although being apparently 'evident'.*
We quote other examples of the most common representations regarding the application of the Second Principle:

(c) $s + s = 200$
$(s + s) : 2 = 200 : 2$
$s = 100$

(d) $s + s = 200$
↓ : 2   ↓ : 2
$s = 100$

(e) $s \times 2 = 200$
$s \times 2 : 2 = 200 : 2$
$s = 100$

We can accept an initial non-conventional representation like an arrow; the passage to a formally more correct writing will be reached step by step.

[22] *In the three preceding situations we applied the principle of cancellation to known quantities; now we apply it also to unknown quantities. After all the steps made, the passage doesn't represent a difficulty for the pupils.*

[23] *We suggest to use also letters which don't have a direct connection with the problem; this is made in order to avoid the phenomenon called* ***semantic persistence*** *(the symbol used as a* 'place card' for the object *instead of being used as a number).*

[24] *In the preceding equation and also in this one, the letter has been deliberately put on* the right. *The aim of this is to avoid the stereotype that the unknown* must *always be on the left.*
It is better not to impose the 'x' for the same reason. The conventional 'x, y ,…' for the unknown; 'a, b, …' for the parameters; etc. must be introduced gradually.

Suitable age related activities	6	7	8	9	10	11	12	13	Comments

Third phase

We propose some verbal problems, which describe situations having a balance as an object. They represent a transitional phase towards more complex verbal problems (**Fourth phase**).

The following considerations are **shared** with the whole class:

- in order to make text lighter, equivalent weights correspond to equivalent nouns;
- the balance is always in a position of equilibrium;
- letter cannot represent different quantities;
- different things are represented with different letters.

13.

On a scale pan there are a packet of candies and a weight of 30 grams.
On the other there is a weight of 110 grams.
Represent the situation in order to find out the weight of the packet of candies.

C = weight of a packet of candies [25]

Possible representations [26]:

C + 30 = 110
C + ~~30~~ = ~~30~~ + 80 factorization and cancellation
C = 80

C + 30 = 110
C + ~~30~~ - ~~30~~ = 110 - 30 first principle and cancellation
C = 80

C + 30 = 110
↓-30 ↓-30 first principle
C = 80

Third phase

Note 3

The real balance is to be used less and less from now on. By now, the pupils must have interiorized the concept of equivalence*; a too prolonged use of the object can create* stereotypes *(the pupils set on a 'concrete' vision of the equation).*

[25] *We suggest to make the pupils write, before the representation, the meaning attributed to the letters used. This should put an end to their unaware use.*

[26] *The pupils must explain all the passages, by describing properties and principles used.*
It is necessary to introduce a gradual 'cleaning' of the writings; at the same time much attention should be paid to the comprehension of their meaning. *These are very delicate phases, during which it is easy to induce wrong concepts and procedural mistakes.*

Suitable age related activities	*6*	*7*	*8*	*9*	***10***	***11***	***12***	*13*	*Comments*

14.

On a scale pan there are 2 tuna fish tins, a packet of spaghetti and a weight of 200 grams.
On the other there are a packet of spaghetti, a tuna fish tin and a weight of 350 grams.
Represent the situation so to find out the weight of a tuna fish tin.

a = weight of a tuna fish tin
b = weight of a packet of spaghetti

$a + \cancel{a} + \cancel{b} + 200 = \cancel{b} + \cancel{a} + 350$
$a + 200 = 350$
$a + \cancel{200} = \cancel{200} + 150$
$a = 150$ [27]

[27] *Another solution (fifth year, primary school, 11 year olds) (see **Comment** 3, **Situation 9**):*

$a + \cancel{a} + \cancel{b} + 200 = \cancel{b} + \cancel{a} + 350$
$a + \cancel{200} - \cancel{200} = 350 - 200$
$a = 150$

Note 4

*Generally in this phase the pupils start wondering which one among the representations used is 'better' than the others. The ones proposed in problem **14** are both good; the one with arrows (for example: in problem **13**) is more efficacious in the initial phases (and often it continues to be preferred by the weaker students). Anyway, it is better to go beyond it, because it interferes with the correct development of formal language.*
*It is advisable that problems of this kind are presented in different moments in time. They should not represent a ‘unique’ occasion for problems ‘different from the usual ones’. Actually, their main aim is to offer a hint for reflection on the different meanings that a letter can have in the mathematical field. This favours a continual progress of the so called **algebraic babbling**.*
On this subject, see Diaries enclosed in the present unit.

Suitable age related activities	*6*	*7*	*8*	***9***	***10***	***11***	*12*	*13*	*Comments*

15.

On a scale pan there are six boxes of cat food, a bedstead and a weight of 1.5 kg.
On the other there are a bedstead, a weight of 2kg, a weight of 0.5 kg and two boxes of cat food.
Represent the situation so to find out the weight of one box of cat food.

x = weight of a box of cat food y = weight of the bedstead

$x + x + x + x + \cancel{x} + \cancel{x} + \cancel{y} + 1{,}5 = \cancel{y} + 2 + 0{,}5 + \cancel{x} + \cancel{x}$ [28]
$4x + 1{,}5 = 2{,}5$
$4x + \cancel{1{,}5} - \cancel{1{,}5} = 2{,}5 - 1{,}5$
$4x = 1$
$4x : 4 = 1000 : 4$ [29]
$x = 250$

[28] *In order to strengthen the pupils' control over the meanings of writings, we suggest to explore different factorizations of numbers. Through successive applications of the principle of cancellation, these factorizations obtain the same results; see, for instance, the equation below:*

$$\ldots + 1 + \cancel{0{,}5} = \ldots + 2 + \cancel{0{,}5} + \ldots$$
or:
$$\ldots + \cancel{1} + \cancel{0{,}5} = \ldots + 1 + \cancel{1} + \cancel{0{,}5} + \ldots$$
or:
$$\ldots + \cancel{1{,}5} = \ldots + 1 + \cancel{1{,}5} + \ldots$$

[29] *Conversion*
1 (kg) → 1000 (g)
used between the third and the fourth passage of the preceding equation simplifies calculations, but can represent a very delicate passage.
It can be convenient (depending on the situation of the class) 'to leave' equations temporarily and concentrate on the numerical meaning of the writing in relation to the situation. Let's suppose that we continue working with kilograms; the preceding equation would end up like this:

…
4x = 1
4x : 4 = 1 : 4
x = ¼ = 0,25

The question to be asked is: What does '0,25' mean in this case?

0,25 of a Kilo
0,25 = 25/100 of a kilo

As 1kg = 1000g, we conclude that:

$$\frac{25}{100} \times 1000 = 250g$$

In the classes of intermediate school, this situation offers an interesting point for reflection on the co-ordination of the possible representations and on their conversion according to the unit of measurement chosen (kg or g).

Suitable age related activities	6	7	8	**9**	**10**	**11**	12	13	Comments

Problems with *indeterminate* solutions. Two examples:

16.

On a scale pan there are two packets of biscuits and a box of chocolates. On the other scale pan a packet of biscuits, 350 grams and other two packets of biscuits. Represent the situation in order to find out:
(a) the weight of a packet of biscuits;
(b) the weight of the box of chocolates.

x = weight of a packet of biscuits
y = weight of the box of chocolates

$\cancel{x} + \cancel{x} + \cancel{x} + y = \cancel{x} + 350 + \cancel{x} + \cancel{x}$ [30]

Whatever the value of 'a', the balance is always in equilibrium.

17.

On a scale pan there are a bottle of milk, a packet of butter and two weights, one of 30 and one of 40 grams. On the other scale pan there are a packet of butter and a weight of 220 grams.
(a) Find out the weight of the bottle of milk;
(b) Can you find out the weight of the butter?

l = weight of the bottle of milk   b = weight of the butter

$l + \cancel{b} + 30 + 40 = \cancel{b} + 220$
$l + 70 = 220$
$l + \cancel{70} = 150 + \cancel{70}$
$l = 150$

The problem doesn't give enough information to find the weight of the butter.

[30] *Another representation (First year, Intermediate school):*

$3x + y = x + 350 + 2x$
$\cancel{x} + \cancel{2x} + y = \cancel{x} + 350 + \cancel{2x}$
$y = 350$

$x = ?$

Suitable age related activities	6	7	8	9	**10**	**11**	**12**	**13**	Comments

Fourth phase

A new series of verbal problems is proposed here; they are very different from the preceding ones because in their texts neither scale pan nor weights appear, but 'a balance is hidden' inside them: they can be solved with an equation. Some of them can also be solved in a non-algebraic way (we will illustrate some examples); in these cases we must use the different solutions for a comparison between the various solving strategies. Also representations through the use of letters can vary a great deal, and they can be the hint for new comparisons and discussions [31].

It is very likely that hybrid writings are proposed; they are forms of algebraic babbling which contribute – thanks to comparison – to an aware formation of the bases of **formal** language. Most of the problem in this phase encloses an iconic support. This is made to favour perception of the elements in equilibrium and consequently of the 'scale pans'.
Probably, the teacher will realise very soon that many pupils tend to look for a **solution** of the problem, and the search for *operations*. The concept of representation must be recovered, by stimulating an approach to the problem that emphasises the analysis of relations among the elements. In this moment, a precise analysis of the text is very important (initially it can be collective) in order to understand which parts allow to identify the 'scale pans' and their contents.
It is better not to use a real balance anymore; if necessary, some situations can be mimed. The pupils should get used to working, being aware of their mental processes. In doing so, we avoid the problem – well known in literature – of the formation of stereotypes, generated by a too prolonged use of metaphors (the scales, bar charts or tree graphs for the solution of problems).

18. Alvise's height

Alvise puts on the table, which is 70 centimetres high, a 30 centimetre stool and he steps on it. Now he is as tall as his father: 1.80 m. Represent the situation so to find out Alvise's height.

Diary 12 (fifth year, primary school, 10 year olds, February)

The comparison is between an 'algebraic' representation:

a = Alvise's height

70 + 30 + a = 180
100 + a = 180 — application of associative property
a + ~~100~~ – ~~100~~ = 180 – 100 — first principle and cancellation
a = 80

and an 'arithmetic' solution:

30 + 70 = 100
180 – 100 = 80

The comparison clarifies the correctness of both solutions, but at the same time if underlines the greater effectiveness of the first, because it is fruitful even if the numerical data are not so simple and the operations are not so immediate [32].

Fourth phase

[31] From a merely mathematical point of view the comparison allows the understanding of the equivalence of expressions - in relation to calculation of a same quantity – elaborated following different paths.

[32] The representations and the solutions proposed by the pupils must be compared and discussed together, to point out that 'a' means 'Alvise's height' and not 'Alvise'. Moreover, the pupils must be aware of the passages of the equation.
A representation of this kind is not unfrequent:

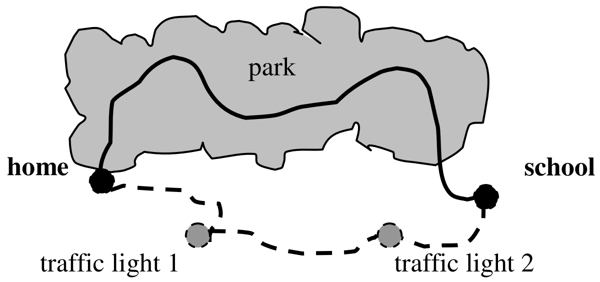
a + 70 + 30 = p

*where 'p' is 'the father's height'. It is a correct way of expressing the relation between Alvise's and his father's height, but is useless for the solution of the problem (to determine Alvise's height). This '**symbol euphoria**' appears in older pupils too, when letters are still '**indicators**' and they haven't yet acquired a useful meaning for the solution of the problem.*
It must be underlined pupils achieve very slowly *the 'right' way of considering relations and of using letters. This achievement requests a great ability of the teacher in directing the pupils' reflections. We want to underline the necessity of dealing with this passage gradually (see algebraic babbling).*

Suitable age related activities	6	7	8	9	10	11	12	13	Comments

19. Fabio and Elena go to school

Elena and Fabio go to school following two different roads: these roads, though, are both 450m long.
Elena walks through the park.
On the other hand, Fabio cycles and passes by two traffic lights and then he still has other 100 metres to cycle.
The first two tracts of Fabio's way are equivalent.
Represent the situation so to find out their lenght.



A correct solution (fifth year, primary school, 10 year olds, February)

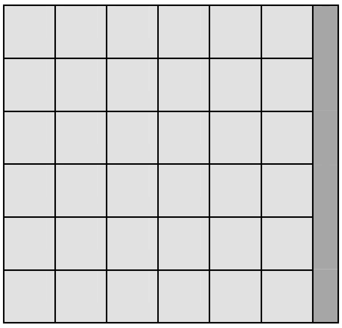
(s = length of each of the equivalent tracts)
s + s + 100 = 450
s + s + ~~100~~ = ~~100~~ + 350 factorisation and cancellation
s + s = 350
2s [33] = 350
2s : 2 = 350 : 2 second principle of the balance
s = 175

[33] *The passage from an additive representation to a multiplicative one is preferred by many pupils.*

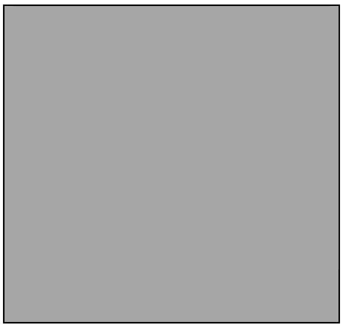
Suitable age related activities	*6*	*7*	*8*	*9*	***10***	***11***	***12***	***13***	*Comments*

20a. The Square and the Park

The spaces represented in the two drawings have the same area of 330 square metres.
Market Square is divided into squared areas: each of them contains a stall.
On its right there is a small garden, whose surface is of 60 square metres.
Represent the situation, so to find out what is the measure of each little square area.



Market Square



Town Park

One solution [34] (Primary school, 10 year olds; intermediate school, 11 year olds)

(a = measure of the square area)

$30 \times a + 60 = 330$
$30 \times a + \cancel{60} - \cancel{60} = 330 - 60$ principle of factorisation
$30 \times a = 270$
$30 \times a : 30 = 270 : 30$ second principle of the balance
$a = 9$

[34] *Between the fifth year in primary school and the first year in intermediate school representations are basically different; it is the spirit itself with which this activity is carried out that leads to creative solutions. They are precious, no matter if they are right or wrong, for the algebraic babbling. Here are some examples:*

- *Correct writings of an additive type:*
- *a + a + …+ a + a + a + 60 = 330*
- *Writings with **syntactic** mistakes (when pupils lose control of the meaning of the process). Two examples:*

$b \times 30 + 60 = 330$
$b \times \cancel{30} + \cancel{60} = \cancel{30} + \cancel{60} + 240$
$b = 240$

$60 + 30b = 330$
$60 + \cancel{30}b + 300 + \cancel{30}$
$60 + 300$
$300 : 6 = 50$

In these situations the teacher's role is essential. He must help the pupils to oppose analysis of the procedures to the frequent 'it's-all-wrong' attitude, only because the result is not the expected one.

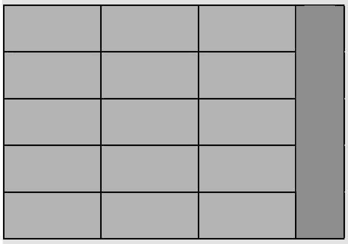
Suitable age related activities	*6*	*7*	*8*	*9*	*10*	*11*	***12***	***13***	*Comments*

Expansion 1

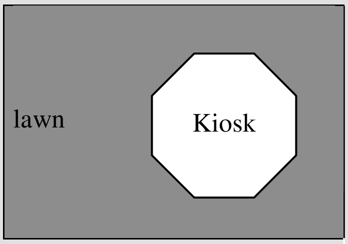
20b. The square and the park.
Two variants for classes of intermediate school, 12 year olds.

Variant 1

The places represented in the two drawings have the same surface.
Market square is divided into equivalent rectangular zones. Each of them contains a stall. Along the right side, there is a small garden, which measures 50 square metres.
The Town Park is formed by a lawn with a surface of 255 square metres and by an octagonal Kiosk covering an area of 15 square metres larger than the area covered by the small garden in the Market square.
Represent the situation in order to find out the surface of each of the rectangular areas.



Market Square



Town Park

One solution

(a = surface of a rectangular area):

$15 \times a + \cancel{50} = 255 + \cancel{50} + 15$ cancellation
$15 \times a = 270$
$15 \times a : 15 = 270 : 15$ second principle
$a = 18$

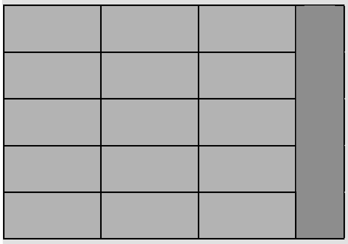
(Continues in the following page)

Suitable age related activities	*6*	*7*	*8*	*9*	*10*	*11*	***12***	***13***	*Comments*

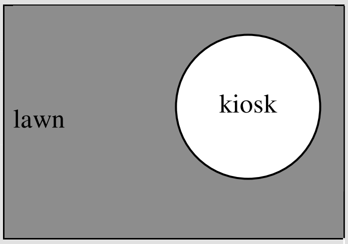
→

Variant 2

The places represented in the two drawings have the same surface.
Market square is divided into equivalent rectangular zones. Each of them contains a stall. Along its right side there is a garden which measures 50 square metres.
The Park is formed by a lawn of 254 square metres, and by a circular kiosk covering a surface, which is triple that of each of the rectangular areas of Market Square.
Represent the situation so to find out the measure of the rectangular areas in Market Square.



Market Square



Town Park

One solution:

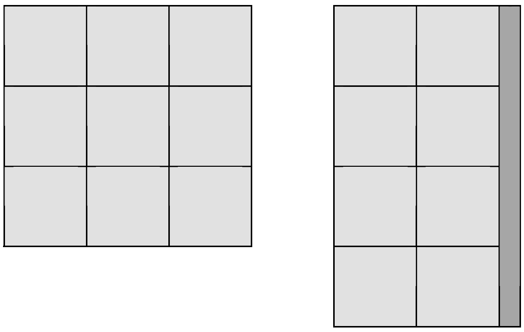
(a = surface of a rectangular area):

$15a + 50 = 254 + 3a$
$15a + \cancel{50} = 204 + \cancel{50} + 3a$ decomposition and cancellation
$12a + \cancel{3a} = 204 + \cancel{3a}$ decomposition and cancellation
$12a = 204$
$12a : 12 = 204 : 12$ second principle
$a = 17$

Suitable age related activities	*6*	*7*	*8*	*9*	***10***	***11***	***12***	***13***	*Comments*

21. The two gardens

> Pinocchio's Garden and Alice's Garden have the same surface and are divided into flowerbeds.
> Alice's Garden has one long flowerbed, whose surface is 70 m^2.
> Represent the situation so to find out the surface of one of the flowerbeds
>
> (their surfaces are all equivalent).



Correct solutions proposed by pupils attending last year in primary school (10 year olds) and first year in intermediate school (11 year olds) [35]:

(a) $9g = 8g + 70$ *[according to pupil g = a, ndr]*
$\cancel{8g} + g = \cancel{8g} + 70$ factorization and principle of cancellation
$g = 70$

(b) $\cancel{a} + \cancel{a} + \cancel{a} + \cancel{a} + \cancel{a} + \cancel{a} + \cancel{a} + \cancel{a} + a = \cancel{a} + \cancel{a} + \cancel{a} + \cancel{a} + \cancel{a} + \cancel{a} + \cancel{a} + \cancel{a} + 70$
$a = 70$

(c) $9 \times p = 8 \times p + 70$ [36]

[35] *As already pointed out, the solutions proposed by the pupils are expressed in a non-algebraic language.*
For instance, two solutions proposed in a fifth year and expressed in verbal language help to understand how to 'solve a problem' doesn't necessarily mean 'to look for operations' nor 'to write calculations'.

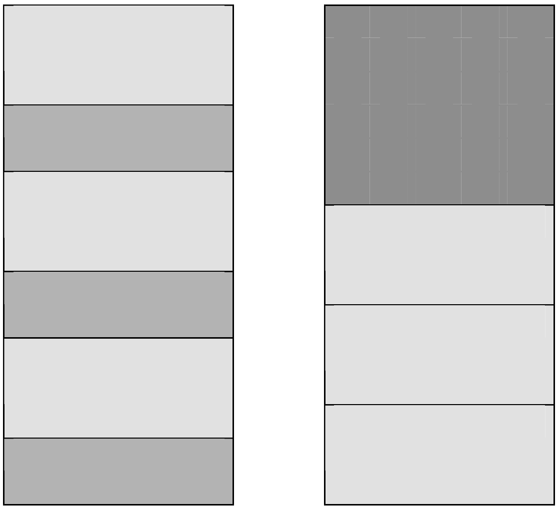
- *«Given that the first garden is formed by nine square flowerbeds and the second by eight square flowerbeds and a long flowerbed. As the two gardens have the same surface, this means that every square flowerbed has the same surface as the long flowerbed: 70 m^2».*
- *«Pinocchio has got 9 flowerbeds. Alice has got 8. 4 pieces worth 1/4 of a flowerbed each* [the pupil imagines the long flowerbed divided into four equivalent parts, ndr*]. If I sum them up, I obtain an 'unknown' flowerbed: hence all the other unknown flowerbeds measure 70 m^2».*

[36] *In this case the pupil didn't know how to continue; probably the writing presents too many syntactic difficulties.*

Suitable age related activities	*6*	*7*	*8*	*9*	***10***	***11***	***12***	***13***	*Comments*

22. The towers

In a kindergarten children have built two towers of the same height using some coloured plastic blocks (same colours correspond to equivalent blocks).
Darker blocks are 48cm high.
Represent the situation so to find out the height of each of the thin blocks in the left tower.



a = height of the middle-size block b = height of the small-size block

$\cancel{3a} + 3b = 48 + \cancel{3a}$ principle of cancellation
$3b = 48$
$3b : 3 = 48 : 3$ second principle of the balance
$b = 16$ [37]

[37] *The phenomenon of semantic persistence is likely to appear, mainly in fifth years (11 year olds) and in first years of intermediate school (12 year olds): pupils write symbols or letters in their equations, and they dispose them in the same order as the blocks in the towers. Here is an example deriving from an 'upside down reading' (although basically correct):*

$B + b + B + b + B + b = B + B + B + 48$

In these same the equation

$$3a + 3b = 48 + 3a$$

can represent the final point of a process of ***negotiation*** *of identity of significance between the writings:*

$$a + a + a = 3a$$
$$b + b + b = 3b$$

(See comments on ***Situation 10*****).***

Suitable age related activities	*6*	*7*	*8*	*9*	*10*	*11*	***12***	***13***	*Comments*

Fifth phase

We propose a new series of verbal algebraic problems; they are more complex than the preceding ones, also because they have no iconic support to help visualise relationships among data.
The teacher can handle the situation with much freedom, employing collective solutions followed by (or alternated with) individual or group solutions, compared and discussed with the whole class. Problems can be chosen and randomly proposed in order to evaluate the growth in autonomy master of the conceptual and procedural aspects. These problems are rich in implications (that will be discussed every time). They are very useful for rereadings of concepts, enrichment of abilities, enlargements, widenings. This phase is very meaningful because it implies a major improvement in the algebraic babbling, forcing the pupils to a progressive sharpening of knowledge. Still, the enrichment is carried out in a very naïve, creative and explorative way, but at the same time this growth brings more and more awareness of rules and conventions. Once again, the main aim is that to search of conditions for representation of the problem, learning to avoid the desperate search for results, which might be defined as a 'result syndrome'.
While facing a problem, first of all pupils should try to understand if there are *conditions of equilibrium*, that is if the problem can be solved with an equation. they must:

- identify the 'scale pans' (the two members of the equation);
- recognise the 'weights' (the known and unknown numerical data):
- represent the situation;
- apply properties and principles and solve the equation.

23. Minerals
The problem contains two unknowns, but one of them, like in the problem of situation **19**, doesn't need to be determined (nor this is possible).

Luigi ordered his collection of minerals on a shelf. He put 31 granites, limestones and fossils in three different boxes. Shortly after his sister Iris accidentally drops the boxes and the minerals spread all over the floor. Iris knows very little about minerals and puts 42 rocks together in a box. She puts the fossils back on the shelf (the only ones she can recognise). Represent the situation in order to find out the number of limestones.

The initial difficulty is that of 'seeing the scale pans', which must be imagined not in space but in *time*: they represent the disposition of minerals *before* and *after* the little accident. The situation can be schematised as follows:

before the accident
31 granites limestones fossils

after the accident
42 minerals together fossils

The minerals are the same, so we can define the equivalence:

31 + c + v = 42 + v [38]

[38] It is always better to make the meanings of the letters used explicit:

c = number of limestones
v = number of fossils

This is not acceptable:

c = limestones
v = fossils

because this writing can favour the idea of letters as 'initials' or as 'place cards'.

The solution implies difficulties that the pupils have already met:

$31 + c + \cancel{v} = 42 + \cancel{v}$ *cancellation*
$\cancel{31} - \cancel{31} + c = 42 - 31$ *I principle*
$c = 11$

24. Games with the computer
The problem is similar to the preceding one: one of the two unknowns cannot be determined.

> Two friends mad about computer games compete on who can get more of them. Initially, Sandro had 22 and a friend gave him a CD with some new games.
> On the other hand, Nicola had 14, and then he found a new game CD, enclosed in a computer magazine; moreover, he received the same CD Sandro got from his friend.
> Now the two friends have the same number of games.
> Represent the situation, so to find out how many games are contained in the CD enclosed in the computer magazine.

In this case there is no 'before' and no 'after'; the 'scale pans' represent the two collections. The situation can be schematised as follows:

Sandro's collection
22 games CD with new games

Nicola's collection
14 games CD from the magazine CD with new games (see Sandro)

The collections have the same number of games, so we can define the equality:

$$22 + x = 14 + x + y$$ [39]

25. The birthday party
This problem has a temporal structure, too: something happened in two different moments: 'before' is equivalent to something 'now'. One of the unknowns appears twice in the left 'scale pan'.

> Roberto invited his friends to his birthday party. To avoid confusion he personalised the plastic glasses with a decalcomania.
> A first group of boys arrives and everyone chooses a glass. Later, a second group of boys arrives. Roberto's mother notices that they are exactly double the number of the first group, and she gives them a glass each.
> At the end of the party, she counts fifteen dirty glasses.
> Represent the situation in order to find out the number of boys in the first group.
> Find out how many friends were in the second group.

The situation can be schematised as follows:

first group	second group	in the end
x glasses	double the glasses	15 glasses

No glasses disappeared, so we can define the equalilty:

$$x + 2x = 15$$ [40]

[39] *We make the meaning of letters explicit:*

x = number of new toys
y = number of toys in the magazine

The solution is not difficult:

$22 + \cancel{x} = 14 + y + \cancel{x}$ *cancellation*
$22 - 14 = \cancel{14} - \cancel{14} + y$ *I principle*
$8 = y$

[40] *An example of a correct solution:*

x = glasses used by the first group

$x + 2x = 15$
$3x = 15$
$3x : 3 = 15 : 3$ *II principle*
$x = 5$

26. The collection of picture-cards
This situation presents a diachronic development, too; there is only one unknown.

> Leonardo started a collection of picture-cards.
> At first he had 23 picture-cards to stick on his album and he received three new packets. Now he has 44 picture-cards.
> Represent the situation so to find out how many picture-cards were in every single packet.

The situation implies a very delicate aspect, pointed out in the following extract from a diary.

Diary 12 (intermediate school, first year, 11 year olds):

> The situation is schematised as follows:
>
> | at first | gift | now |
> |---|---|---|
> | 23 picture-cards | 3 packets | 44 picture-cards |
>
> Pupils stop here because they think they have two unknowns called in a jargon that doesn't help them 'the packets' and 'the picture-cards' [41].
> The analysis of the draft points out that in fact 'the packet' is not an unknown, but 'the container of the unknown' (the number of picture-cards). In this perspective we can write:
>
> x = number of picture-cards in a packet
>
> Hence, we understand that the mathematical meaning of '3 packets' is:
> 'number of picture-cards in a packet' *multiplied by* 'number of packets'
> √ The pupils are asked to represent la situation for **Brioshi.**
>
> (a) $3 \times x$ (b) $x \times 3$ (c) $3x$
> The discussion leads to a choice for representation (c).
> Now we can define the equality:
>
> $$23 + 3x = 44$$ [42]

27. The camping
This is a problem with a single unknown. Once again the pupils meet the same difficulty: the tent must be seen in relation to the unknown number of its sleeping places.

> Rita and Gabriella went on holiday in two scout campings.
> Rita's camping his formed by her 6-place tent and other 4 tents, different from hers, but all alike.
> In Gabriella's camping there are 54 boys and girls, just like in Rita's.
> Represent the situation so to find out how many people can sleep in each of the 4 identical tents in Rita's camping.

The situation is schematised as follows:

Rita's camping		Gabriella's camping
1 six-place tent	4 n-place tents	54 people

n = number of people in a tent.
We write the equality:

$$6 + 4n = 54$$ [43]

[41] *The situation here described is very common; although with a correct algebraic background, in the second and third year of intermediate school this difficulty may not occur.*

[42] *Two examples of a correct solution:*

$23 + 3x = 44$
$\cancel{23} + 3x - \cancel{23} = 44 - 23$ I principle
$3x = 21$
$3x : 3 = 21 : 3$ II principle
$p = 7$

$23 + 3x = 44$
$23 + 3x = 23 + 21$ decomposition
$\cancel{23} + 3x = \cancel{23} + 21$ I princ. (cancellat.)
$3x = 21$
$3x : 3 = 21 : 3$ II principle
$p = 7$

[43] *An example of correct solution:*

$6 + 4n = 54$
$\cancel{6} + 4n - \cancel{6} = 54 - 6$ I principle
$4n = 48$
$4n : 4 = 48 : 4$ II principle
$n = 12$
If the concept of 'container of the unknown' is no clear, writings such as

$t + 4t = 54$

may appear, in which 't' indicates the object *and not the number.*

> ### *Note 5*
>
> *It better to sum up all the significant algebraic steps presented in the solution of problems **26** and **27**:*
>
> - *We give a name to the number of picture-cards in a packet (e.g.: f) or of places in a tent (e.g.: p);*
> - *By using the link between the number of packets (or of tents) and that of picture-cards (or of places) we make the total number of packets ($3 \times f$, o $3f$, o $f \times 3$) and that of the tents ($6 \times p$, o $6\,p$, o $p \times 6$) clear.*

Suitable age related activities	*6*	*7*	*8*	*9*	*10*	*11*	***12***	***13***	*Comments*

28. Federica's pocket money

The difficulty in this case is represented by the fact that the unknown appears in both members; identifying the two terms (the 'scale pans') is more laborious.

> On Wednesday Federica did some housework and her mother rewarded her with some money.
> On Thursday Federica helped her and she was really good, and her mother gave her 12000 lire. By adding this pocket money to that received on Wednesday, Federica obtains as a result triple the money she had before.
> Represent the situation so to find out how much money Mum gave to Federica on Wednesday.

If necessary, teachers can help to understand the situation better, by asking questions such as «What happened on Wednesday?» «And then on Thursday?» «And then how much money does Federica have?» and so on.

The situation is easier to understand if represented step by step:

ney gained on Wednesday + 12000 = money gained on the whole
ney gained on the whole = triple the money gained on Wednesday

These can be translated into mathematical language with a system of equations:

(i) $s + 12000 = t$
$t = 3s$

The substitution leads to a single equation:

(ii) $s + 12000 = 3s$ [44]

29. The phone cards

The mathematical structure is the same as that of the preceding problem.

> Two brothers, Susanna and Martino, collect phone cards: When they compare their collections, they realise they have the same number of cards.
> Susanna has an envelope with some French cards and 33 English cards.
> Martino counts his own cards and he realises that they are exactly four times as many as his sister's French cards
> Represent the situation in order to find out how many cards Susanna has got in the envelope.

An example of a correct solution

(n = number of French cards):

$n + 33 = 4n$
$n + 33 - \cancel{n} = 4n - \cancel{n}$ I principle
$33 = 3n$
$33 : 3 = 3n : 3$ II principle
$11 = n$ [45]

[44] *An example of a correct solution:*

$s + 12000 = 3s$
$\cancel{s} - \cancel{s} + 12000 = 3s - s$ *I principle*
$12000 = 2s$
$12000 : 2 = 2s : 2$ *II principle*
$6000 = s$

We can stop at this writing, leaving the unknown on the right part of the equal sign_but it is worth while remind the pupils that, as the two parts of the equation are symmetrical, we can also write:

$s = 6000.$

[45] *The habit of putting the letter on the left ends up prevailing. We can justify this on a linguistic basis: it is more common to say*
'the value of the unknown is 11'
rather than
'11 is the value of the unknown'
exactly as in the spoken language we say
''the colour of the hat is red'
rather than
'red is the colour of the hat'.
Obviously, both in the linguistic and mathematical contexts the two expressions are equivalent.

Note 6

*In the problem of situation **28** the passage from (i) to (ii) implies the substitution of 't' with '3s'. It is a logical passage, which might not be so simple to understand for some pupils. A strategy can be applied to help comprehension through the metaphor of 'the game of change' (see Malara, 1999).*

30. Youth games

In a school during the swimming competition of the Youth games both girls and boys have enrolled, but the number of boys is one third of that of girls.
The number of participants in the swimming competition is the same as that of the participants to the race: in both competitions there are 52 competitors.
Represent the situation in order to find out how many girls have enrolled for the swimming competition. [46]

Expansion 2

With students of the third year (intermediate school) a rudimentary system might be built:

$m + f = 52$
$m = \frac{1}{3}f$

or

$m + f = 52$
$f = 3m$

and then operate some substitutions.
It is clear that for most of the students the second system is less complicated than the first, although the first represents the situation described in the text with higher fidelity.

31. The search for fossils

Two separate groups of students are busy looking for fossils among the rocks of a river.
The students in the second group are less lucky than the first, because they collect exactly half the fossils the first group has found.
After a few hours the two groups gather and put the fossils in common. On the whole, 36 fossils have been found.
Represent the situation in order to find out how many fossils have been found by the luckier group.

An example of a correct solution:

$f + f + f = 36$
$3f = 36$
$3f : 3 = 36 : 3$
$f = 12$

Here is an interesting rudimentary system composed by two operations with two unknowns (13year olds):

$36 = A + B$; $A = 2B$
$36 = 2B + B$
$36 = 3B$
$3B = 36$
$3B : 3 = 36 : 3$
$B = 12$ [47]

[46] *This problem can be represented in many ways, depending on the skills students have acquired with fractions and on the point of view from which the relations between quantities are considered. Here are some commented protocols by pupils of fifth year, primary school (12 year olds):*

(a) $f + f/3 = 52$

Difficult path unless one can operate properly with fractional numbers; often the pupils who propose it get stuck or they change strategy:

(b) $f + f : 3 = 52$
This path is even more complex; frequently it induces the error of applying the second principle and dividing both members by 3.

(c) $3m + m = 52$
$4m = 52$
$4m : 4 = 52 : 4$
$m = 13$
reversing the relation is often the best solution, even though it is not the most frequent; it is interesting 'to provoke' it (this device really simplifies the solution).

[47] *Some examples of correct equations and of strategies autonomously activated to solve them:*

(d) The author writes
$A + A : 2 = 36$
he doesn't know how to continue and he starts again:

$\frac{1}{2}A + \frac{1}{2}A + \frac{1}{2}A = 36$
$\frac{3}{2}A = 36$
$\frac{3}{2}A : 3 = 36 : 3$
$A = 12$

(e) According to complexity, pupils alternate multiplicative and additive representations:

$2f + f = 36$
$f + f + f = 36$
$3f : 3 = 36 : 3$
$f = 12$

Suitable age related activities	*6*	*7*	*8*	***9***	***10***	***11***	***12***	***13***	*Comments*

Evaluation file

Optimum skill: the pupil uses letters consciously to:
1) represent the problematic situation (equation) in a correct way,
2) solve the equation by applying properties and principles correctly.

The pupil

1: uses **iconic** symbols
- semantically significant (segments for roads, square for flowerbeds)
- semantically non significant (rectangles, triangles, asterisks, and so on)
- redundant, detailed, unnecessary, …

in order to set up a representation
- correct (**pseudo-equation**: see the *Comment* column)
- as an effective though 'external' support for a solution
- as a correct but 'dumb' support of meanings
- wrong
- doesn't use any representation

2: uses **literal** symbols
- semantically significant ('f' for 'flowerbed', 'b' for 'bush', and so on)
- semantically non significant (any letters).

in a way that is
- correct (he sets up the equation)
- confused
- wrong (unaware use)
- doesn't use any letters

in order to set up an equation
- clear and correct as an **equivalence** between quantities
- not completely correct
- approximately correct
- wrong
- doesn't know how to set up an equation

3: applies a reasoned decomposition of a number
- in a highlighted and correct way
- in a highlighted way, but with minor mistakes in calculations
- in a confused way (sometimes he highlights and sometimes he doesn't)
- in a wrong way
- he doesn't know how to apply it

4: makes a cancellation explicit
- in a clear and correct way
- correctly, but he doesn't highlight the decomposition of the number
- in a partially correct way
- in a wrong way
- he doesn't use cancellation

5: uses brackets
- correctly
- incorrectly

Evaluation file

This is a hypothesis still to be verified; it represents a trace for observation of the pupils, which is based on the experiences in class.

1

- *This entry encloses also some naïve representations, alien to algebraic language (they are likely to appear mainly in the initial phases of activity with scales).*
- *With the term 'pseudo-equation' (borrowed from the researcher Da Rocha Falcâo) we mean a spontaneous representation made by the pupil. Although it is not an orthodox equation, it highlights a rudimentary algebraic thought. The pseudo-equation is didactically very important with younger pupils.*
- *It is important to notice the possible presence semantic persistence).*

2

It is not advisable to draw the pupils' attention on the use of particular letters; the conventional aspects of algebraic language become gradually clear.

5

We refer here to the use of round brackets in cases such as:

$$(a + a + a) : 3$$