A THEORETICAL CONSTRUCT TO ANALYZE THE THEACHER'S ROLE DURING INTRODUCTORY ACTIVITIES TO ALGEBRAIC MODELLING

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Abstract

In this work we will introduce a theoretical construct that we have elaborated as a tool for the analysis and the interpretation of the teachers' actions during class activities which are aimed at fostering an aware learning of the use of algebraic language as a thinking tool. Through the analysis of an excerpt of a class discussion concerning introductory activities to algebraic modelling, we will show how this construct could provide "transparent" indicators to highlight the effectiveness of the teachers' actions during class interaction.

1. INTRODUCTION

The idea of an early approach to algebra, with a strong focus on generational activities (see Kieran 1996), is widespread and consolidated (Carpenter & al. 2003, Kaput & Al. 2007, Cai & Knut 2011). In order to overcome the well-known difficulties usually faced in the study of the formal aspects of algebra, students are introduced to algebraic modelling, both in realistic and mathematics contexts, but also to the deduction of properties by means of algebraic language (Chevallard 1989/1990), with the aim of giving them the opportunity to develop both a deep awareness of the origin of algebraic objects and an effective symbol sense (Arcavi 1994). Forerunners in this field are the studies carried out by Bell & Al. (1987), who promote a teaching aimed at guiding students through the essential algebraic cycle "represent, manipulate and interpret".

Starting from the 90s, these ideas are developed by research together with a new vision of the teaching of arithmetic, caracterised by a focus on relational aspects and meta-level activities, aimed at making students control the properties subtended to arithmetical equalities in order to create a connection between arithmetic and algebra (Bell 1996, Kieran 1996, Lincevski 1995). The most recurrent didactical activities which are proposed concern, besides problem solving, the generalization and modelling of: figural and numerical sequences, functional relationships in realistic contexts, numerical games for the identification and justification of the subtended mathematical aspects. In the first decade of 2000 many research studies are devoted to the implementation of these activities at school (also at the primary level), documenting students' successful learning and their development of new and more productive attitudes. However, only few of these studies consider the role played by the teacher together with the problem of teacher education (see for example Carpenter & Franke 2001, Blanton & Kaput 2001a&b).

Our research studies can be inserted in this theoretical frame; at the beginning they were devoted to the planning of innovative didactical paths in arithmetic and algebra (grades 4-8) to be implemented through a socio-constructive approach and they were

caracterised by a strict cooperation with the teachers, usually teacher-researchers (Malara & Navarra 2003).
The positive effects of these activities on students'learning suggested us to carry on with these experimentations, involving a larger number of motivated and experienced teachers. These studies enabled us to highlight two main gaps: a gap between teachers' declared conceptions and the hidden ones displayed by their behaviours in the classes; and a gap between the theoretical assumptions they shared with researchers and their actual practice (Malara 2003). Closely intertwined with class experimentations, new teacher education activities were therefore introduced in our project. Our main references in the planning of these activities were, besides the Italian studies, the research developed by Mason (1998) and Jaworski (2003), who suggest to foster teachers' development of different levels of awareness through the activation of joint critical-reflection practices (Malara 2008). Thanks to this experience we designed and implemented new tools and methods to be used in communities of inquiry (Jaworski 2006), which have proved to be effective ways of fostering teachers' real professional development (Cusi, Malara & Navarra 2011).
In particular, our studies on the use of algebraic language as a tool for thinking in the construction of proofs (grades 9-10) enabled us to define a theoretical construct (Cusi & Malara 2009, Cusi 2012) which highlights the specific features of a teacher who poses him/herself as a "*model of aware and effective attitudes and behaviours*" (in the following M-$_{AE}$AB).
In this paper we will introduce the M-$_{AE}$AB construct in the theoretical frame which constitutes its background and we will show its effectiveness as a "theoretical lens" for the analysis of the role played by the teacher during class activities aimed at the introduction of algebraic modelling. Moreover, we will propose some reflections about the use of this construct in teacher education activities as both a diagnostic tool in the analysis of class processes and a tool for teachers' self-reflection on their own teaching.

2. THE M-$_{AE}$AB CONSTRUCT FOR THE ANALYSIS OF THE TEACHER'S ROLE

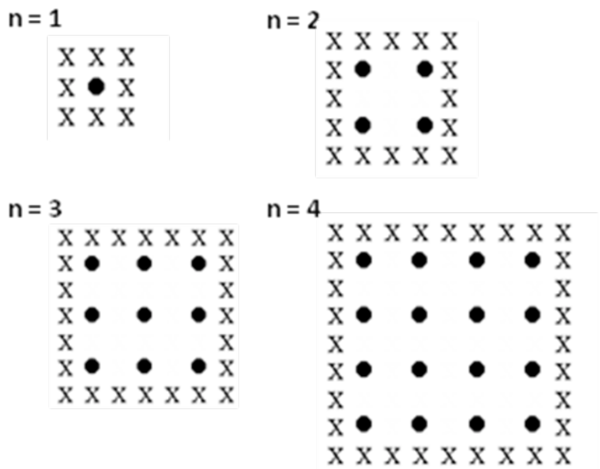
The theoretical frame within which the M-$_{AE}$AB construct has been developed is constituted by two threesomes of components. The *first threesome* refers to the theoretical components we identified for the analysis of the development of thinking processes through algebraic language: (a) the model of didactic of algebra as a thinking tool proposed by Arzarello & Al. (2001), who, in particular, highlight the essential role played by the activation of conceptual frames and appropriate changes from a frame to another for a correct interpretation of the algebraic expressions which are progressively constructed; (b) the idea of anticipating thought developed by Boero (2001), who introduces it as a key-element in the "game" transformation-interpretation, which is typical of the processes of construction of reasoning through algebraic language; (c) the theoretical analysis proposed by Duval (2006), who identifies in the coordination between different representation registers a critical aspect in the development of learning in mathematics.

The *second threesome* of components is related to our theoretical framework of approach to the study of the teaching-learning processes and of the role played by the teacher. The first component is Vygotskian: we, in particular, refer to Vygotsky's stress (1978) on the importance of a teaching aimed at expanding students' zone of proximal development in order to stimulate, thanks to their interaction with the teacher or with more expert classmates, the activation of internal learning processes associated to a higher level of mental development.
The second component draws its inspiration from the work carried out by Leont'ev (1978), who stresses the importance of making students increase their awareness about the meaning of the processes they activate during class activities in order to foster their learning. These basis enabled us to outline an idea of teacher's action that we could develop referring to some aspects of the cognitive apprenticeship model, our third component. This model, introduced by Collins & Al. (1989), drawn its inspiration from an idea of learning as an "aware" apprenticeship and pursue the objective of "making thinking visible", through the activation of teaching methods which give students the opportunity of observing, discovering or even inventing the experts' strategies in the same context in which they are worked out.
In our work, in particular, we refer to two sets of typical methods of cognitive apprenticeship: (a) modeling, coaching and scaffolding, aimed at helping students acquire skills through processes of observation and guided practice; (b) articulation and reflection, related to metacognitive objectives and aimed at helping students achieve a conscious control of their own problem-solving strategies. [1]
We think that the cognitive apprenticeship paradigm offers suitable reference points for the study (planning, implementation and analysis) of teaching-learning processes aimed at fostering an effective use of algebraic language as a thinking tool. The 'games' of coordination between different linguistic registers and of interaction between the syntactical level, the interpretative level and the level of activation of anticipating thoughts, which can be automatically set up by an expert, should be "made visible" to novices in order to make them acquire and understand their meaning. Since we believe that a real acquisition of knowledge always requires a good control of the meaning of the processes which lead to it, our hypothesis is that the teacher, during class interaction, should adopt and make visible specific attitudes and behaviours in order to guide his/her students, through a process of cognitive apprenticeship, to the acquisition of the same attitudes and behaviours, which enable them to progressively develop the competences and awareness necessary to carry out advanced tasks, such as the construction of proofs through algebraic language.
Thanks to the analysis of the role played by the teachers who participated in our experimentations (examples of this analysis can be found in Cusi & Malara 2009 and in Cusi 2012) and the comparison between their different approaches, we were able to clearly highlight how unsuitable choices can lead to a missed acquisition of competences and awareness by students. Moreover, we singled out, in contrast with these unsuitable approaches, the specific characteristics of a teacher who is able to act in order to both "make thinking visible" and, at the same time, guide his/her students to the development of an awareness of the meaning of the activated processes. We

chose to objectify the profile of this kind of teacher through the $M_{AE}AB$ theoretical construct, identifying its distinguishing features, which can be placed in some fundamental behavioural categories of the cognitive apprenticeship. First of all, this kind of teacher poses him/herself: (a) as an "*investigating subject*", stimulating in his/her students an attitude of research towards the problem being studied, and as a *constituent part* of the class in the research work being activated; (b) as a *practical/strategic guide*, sharing (rather than transmitting) with his/her students the adopted strategies and the knowledge to be locally activated; (c) as a provoker, who stimulates the construction of the key-competences for the development of thought processes by means of algebraic language, playing the role of an *"activator" of processes of generalization, modelling, interpretation and anticipation*.
These roles that should be performed in the class can be placed in the *categories of modeling and coaching*. They require the teacher to carry out the activities posing him/herself not as a "mere expert" who proposes effective approaches, but as a *learner* who faces problems with the main aim of making the hidden thinking visible, highlighting the aims, the meaning of the strategies and the interpretation of results.
Other important features of the profile of a teacher as a $M_{AE}AB$ are the following: (d) he/she poses him/herself as guide in controlling the meaning of the constructed algebraic expressions both at the syntactical and at the semantic level, with the aim of maintaining a *harmonized balance* between these two aspects; (e) he poses him/herself as a *reflective guide* in identifying effective practical/strategic models during class activities (he/she also stimulates reflections on the effective approaches carried out during class activities in order to make students identify them as models from which they can drawn their inspiration); (f) he/she poses him/herself as an *"activator"* of both *reflective attitudes* and *meta-cognitive acts*, with the aim of stimulating and provoking *meta-level attitudes*, with a particular focus on the control of the global sense of processes.
These last distinctive characteristics, that can be placed in the *categories of articulation and reflection*, refer to a different role played by the teacher: he/she must also be a point of reference for students to help them clarify salient aspects at different levels, with an explicit connection to the knowledge they have already developed. Playing this role fosters, in tune with the ideas developed by Leont'ev, students' development of a real awareness of the meaning of both the class activities and the learning processes themselves.

3. THE ANALYSIS OF A CASE

The following excerpt refers to the initial part of a discussion conducted in a first class of lower secondary school (grade 6). The specific activity was proposed at the end of an introductory path to the algebraic modelling of figural sequences. The problem situation was adapted from the Pisa task usually named as "the apple trees". The characterizing feature of this task is the combination of the figural and verbal registers with the aim of fostering generalization and the algebraic formalization of the relationship between the number of apple trees and the number of conifers in the different possible configurations. In order to simplify the problem situation and to help students in its exploration and in making the identified relationships explicit, tables were introduced together with the requirement of specific argumentations. Moreover, in order to make the problem situation more engaging for students, the worksheet was graphically enriched. Due to space limitations we do not present the original worksheet, but only the proposed patterns and the first questions.

	Below you can find the patterns which represent the disposition of apple trees and conifers in relation to the number (n) of the rows of apple trees. 1) After having carefully observed the patterns, what can you say about the disposition of apple trees and conifers in the different cases? 2) Try to reproduce, through a drawing, the disposition of apple trees and conifers when n=5. Motivate your answer. 3) Explain how you can find the number of apple trees if you know the number of rows.

In the class discussion that we propose, the teacher tries to guide her students to the exploration of the number of conifers and apple trees in the different patterns.
The left column of the following table contains the excerpt of the first part of the discussion (T stands for the teacher, while the other alphabetical letters stand for the different students who take part in the discussion). In the right column we propose an analysis of the teacher's interventions with reference to the theoretical construct $M_{AE}AB$. The main aim of this analysis is to show how this construct could help in highlighting the effectiveness (or the inappropriateness) of teacher's interventions also during class discussions which refer to introductory activity to algebraic modeling.

Class discussion excerpt	**Analysis of T' interventions through the $M_{AE}AB$ construct**
The class exploration starts with T's request of reproducing the patterns on the workbook while she is doing the same at the blackboard.	
1. T: What did you check, while I was drawing on the blackboard, to exactly reproduce the disposition of apple trees and conifers?	T poses herself as an ***investigating subject***, stimulating an attitude of research towards the problem. Moreover she simulates an ***attitude of sharing***.
2. M: (I checked) how many conifers there are on each side. 3. RB: I checked how much the number of apple trees increases passing from one drawing to another and how many conifers there are on each side.	

4. A: (I checked) how many apple trees there are in all. 5. G: I checked the rows. 6. K: In the first drawing there are 9 conifers.	
7. T : In the first drawing there are 9 conifers. How did you determine the correct number of conifers, K?	T poses herself as a ***reflective guide***. When K looks at the total number of the conifers and makes a mistake, T does not express any judgment. On the contrary, she intervenes to turn K's attention to the counting strategies he adopted in order to prompt a correct attitude of inquiry and to foster a self-correction.
8. K: I did 3... 3… I got wrong.	
9. T: Try to explain that.	T poses herself as an ***activator of metacognitive acts***: she fosters an attitude of enquiry, encouraging K so that he can be able to make his thoughts explicit.
10. K: They are 8. I considered 3 at the beginning, on the first side, then I added 2, then 2 on the other side, and then 1.	
11. T: Eight. Good, K! And how many apple trees are there instead, G?	T encourages again the students and poses herself as a ***practical-strategic guide***, making them focus on the first configuration and re-directing the inquiry towards the identification of the interrelation between the number of conifers and the number of apple trees.
12. G: One!	
13. T: Let's explore the other representations as well. How many conifers are there in the second representation?	T poses herself as a ***participant***, constituent part of the class group, and as a ***strategic guide***, drawing students attention toward the second configuration.
14. GF: 8 multiplied by 2. Two is the number of the rows. Therefore 16.	
15. T: A said that he would have wanted to know how many apple trees are exactly in the drawing.	T poses herself as an ***activator of reflective attitudes***, trying to focus students' attention to a comparison between the different cases with the aim of making them highlight a correlation between the number of conifers and the number of apple trees.
16. GP: In this one there are 4 (apple trees)	

17. M: I noticed that the number of rows is equal to the number of apple trees in the rows.	
18. T: What would you say about M's observation?	T does not judge M's observation and ask the other students to examine it, posing herself as a ***reflective guide***, with the aim of both stimulating reflections on the different approaches proposed and making them explicit.
19. A: It's right. When n=2 there are two apple trees in every row. 20. G: So, in order to calculate the number of apple trees in the enclosure we should multiply the number of the rows by the number of trees in every row. 21. K: I didn't understand anything.	
22. T: The observations actually overlapped.	When K declares his doubts, T poses herself again as a ***participant***, stimulating the class in order that the different proposed observations could be better made explicit. In this way she fosters the sharing of knowledge and poses herself as an ***activator*** of both ***reflective attitudes*** and ***metacognitive acts***.
23. G: I meant to say that in this case, K, in order to calculate the number of apple trees you must take the number of apple trees in every row and multiply it by the number of rows. Therefore two multiplied by two. 24. M: That is you must multiply the number of rows by itself because the number of apple trees is equal to the number of rows.	
25. T: So let's see if I am able to understand. What do "the number of rows" and "the number of apple trees in every row" mean?	Instead of evaluating G and M's observations, T poses herself as a ***reflective guide***, asking students' to clarify the meaning of some terms, with the aim of "making their thinking visible". In this way she provokes "explicitations" and stimulates reflections on the different approaches, posing herself as an activator of both ***reflective attitudes*** and ***metacognitive acts***.
26. A: The rows are those (he points at the drawing) ... that is the	

number of rows, how many rows there are. The number of apple trees is how many apple trees there are in every row. 27. K: I have understood!	
28. T: I have understood now. Thanks, M. So how can we write this number 4 which stands for the number of apple trees?	T ***stimulates*** and ***provokes*** the ***construction*** of key-competences for the development of thought processes by means of algebraic language, posing herself as an ***activator of interpretative processes***, gradually stimulating the activation of correct conversions from the verbal to the symbolic register.
29. Group of students: 2 multiplied by 2!	
The discussion continues with the analysis of the number of the conifers in every pattern and the following identification of the symbolic expressions which represent the relation between the number of apple trees and the number of conifers in every configuration and the number of rows. Lastly it ends with a naive study of an inequality in order to determine in what cases the number of the apple trees exceeds the number of the conifers.	

4. FINAL REMARKS

Through the theoretical lenses we adopted for the analysis of the previous excerpt, it was possible to highlight an effective action of the teacher, in tune with our theoretical frame of reference, caracterised by a specific focus on the strategies aimed at making students control their thinking processes and develop an awareness about the meaning of the performed activities. We also tested, with good results, the use of the $M_{AE}AB$ construct in the analysis of other less sound discussions, concluding that its characteristic components can be considered "transparent" indicators to highlight the effectiveness (or not) of the teacher's action. This construct seems therefore to be an effective diagnostical tool in the analysis of the quality of the teacher's management of class activities aimed at fostering an aware learning of algebraic language.

Moreover, we believe that the $M_{AE}AB$ construct could be also a useful tool to promote teachers' reflection on their own practice. In tune with Mason's idea of teaching as "educating awareness" (1998), we think that making the teachers analyse their class processes through specific theoretical lenses could provoke what Mason defines "shifts of attention", which play an essential role in fostering the development of new awareness and hence in determining an effective teaching. We believe indeed that these activities could allow teachers to perform their first "guided" reflective practices, receiving and afterwards interiorizing the necessary stimulus for the construction of their own models for reflection, to which they can refer everytime they have to analyse their practice. In the future we intend to test this hypothesis referring to the $M_{AE}AB$ construct in the work with both pre-service and in-service teachers, proposing it to them as a tool for self-analysis.

NOTES

1. *Modeling* require that an expert performs a task externalizing the internal processes in order to make students observe and build a conceptual model of the processes that are required to accomplish it; *coaching* consists of observing students while they carry out a task to offer them hints, scaffolding, feedback; *scaffolding* refers both to the supports the teacher provides to help the students carry out a task and to the gradual removal of the same supports (named *fading*) in order to let the students autonomously perform the task. *Articulation* involves the methods applied to make students articulate their knowledge, way of reasoning and problem-solving processes; while *reflection* involves enabling students to compare their own problem-solving processes with those of an expert or of another student, so that they ultimately could be able to compare them with an internal cognitive model of expertise.

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