Which algebraic learning can a teacher promote when her teaching does not focus on interpretative processes?

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In this paper we will focus on the effects of teachers' lack of the algebraic knowledge and sensitiveness that are necessary to effectively control the plurality of levels of interpretation involved within activities aimed at making students aware of the meanings associated to their use of algebraic language. Our analysis of class discussions conducted by a group of middle-school teachers involved in university training courses enabled us to highlight profound gaps, between modeling and interpretation, that prevent them from becoming aware of what it is important to stress when facing this kind of activities with students.

Keywords: interpretative processes, algebraic thinking, algebraic teacher knowledge for teaching, teacher education.

FOCUS ON INTERPRETATIVE ASPECTS IN THE TEACHING OF ALGEBRA: KEY ROLE OF THE TEACHER

Since the development of the first studies on the problems related to the learning and teaching of algebra, many researchers reject the previous widespread idea that students' difficulties are mainly related to the complexity of algebraic syntax (see for instance Ursini, 1990; Kieran, 1992). The focus has therefore shifted on students' control of the meanings associated to their use of algebraic language. Different studies stress the importance of stimulating them with the aim of fostering their aware use of symbols as tools to represent, communicate, generalize, solve problems, develop reasoning (Arcavi, 1994; Arzarello, Bazzini, & Chiappini, 2001; Kieran, 2004). Bell (1996) states, in particular, that if students are given the possibility to have experience of the use of algebraic symbolism as a tool to express regularities and represent relationships, they could be guided through what he calls the "essential algebraic cycle", characterised by three main typical algebraic activities: to represent, to manipulate and to interpret.

In this work we focus on the third component of the essential algebraic cycle: the interpretative processes that are typical of algebraic activities. When we use the term '*interpretative processes*' we refer both to Duval's (2006) idea of conversions between different representation registers (in the case of algebra, from symbolic to verbal register and viceversa, but also from graphic, to symbolic, to verbal) and to Arzarello, Bazzini and Chiappini (2001)'s model for teaching algebra as a *game of interpretations*. These last researchers highlight, in particular, how the activation of conceptual frames (defined as an "organized set of notions, which suggests how to reason, manipulate formulas, anticipate results while coping with a problem") and the changes from a frame to another and from a knowledge domain to another could represent fundamental steps in the activation of interpretative processes.

The problem of algebraic modeling and of the coordination between verbal and algebraic language have been a central focus of research since the eighties. A paradigmatic study is the one developed by Clement (1982), who documented erroneous approaches developed by science-oriented college students in facing simple kind of algebra word problems. As regard to this problem, Smith and Thompson (2007) suggest that students' difficulties with algebra result not only from algebra curricula that lack meaning and coherence, but also from elementary curricula that fail to develop students' abilities to reason about relationships. This observation has been shared by different research studies developed in the realm of Early Algebra. Blanton and Kaput (2005, 2011), for example, highlight the crucial role played, in elementary school mathematics, by the algebraic reasoning embedded in finding, describing, justifying and symbolizing mathematical relationships between quantities, advocating a functional approach to algebra in the elementary grades. This is in tune with Kieran (2004), who refers to the shift of focus from the calculation of numerical answers to the relationships between quantities as one of the fundamental adjustments that students should make in their transition from arithmetic to algebra. This vision is in tune with the approach that we have developed within the ArAl Project (see for instance Cusi, Malara, & Navarra, 2011), aimed at proposing a relational and linguistic approach to Early Algebra and also meant to constitute an integrated teacher education program.

A recent study by Magiera, van den Kieboom and Moyer (2013) has highlighted that pre-service middle school teachers demonstrated a rather limited ability to recognize the full potential of algebra-based tasks to elicit algebraic thinking in students. They suggest, therefore, that there is a need of making teachers understand the contexts in which the various features of algebraic thinking might arise in order to enable them to effectively engage students. In tune with these results, Strand and Mills (2014), in their survey of research literature on prospective elementary school teachers' knowledge of algebra, state that it is well documented that these teachers tend to struggle to effectively interpret and use algebraic symbols (even those that they have produced themselves), to interpret graphical representations and to make connections between representations.

Our experience as teacher educators enabled us to observe similar problems also in the Italian context. Although research results about the importance of guiding students through the whole essential algebraic cycle have been widespread, Italian teachers are frequently not used to focusing on interpretative aspects. As a result, formalism and application dominate. This requires us to shift the focus on the teacher knowledge of the interpretative aspects connected to algebraic thinking and on the effects of this knowledge on students' learning. A framework to identify the knowledge for teaching school algebra has been developed by McCrory, Floden, Ferrini-Mundy, Reckase and Senk (2012), who stress the role played by teachers' capability of selecting, applying, and translating among mathematical representations and of making connections between manipulatives and mathematical ideas explicitly. But this way of behaving in the class must be educated. In order to foster teachers' overcoming of these difficulties and their effective approach in guiding students in the transition from arithmetic to algebra, in our in-service training courses for middle-school teachers we propose specific laboratorial activities focused on this theme.

In this paper we will present the main results from the analysis of the first of these activities, which was aimed at making trainee-teachers (in the following TT) directly experience how (and if) they are able to foster students' activation of interpretative processes during whole class discussions about a task involving proportional relations. In particular, we will highlight how this analysis enabled us to document the effects, in TT' way of conducting class processes, of their lack of the specific knowledge that could favour students' development of interpretative attitudes.

METHODOLOGY OF WORK WITH TRAINEE-TEACHERS

The laboratorial activity we present was the first one proposed to a group of 58 middle-school (grades 6-7-8) TT involved in a training course aimed at making them achieve a teaching qualification. This course, which lasted six months, was specifically devoted to temporary teachers that have been working in school for at least three years. In Italy, in fact, also people who do not have a teaching qualification could work as teachers in school. As regards, in particular, the middle-school context, mathematics is mainly taught by teachers who do not have a mathematics background (their degrees could be in biology, chemistry, natural science, geology…).

During the whole training period, the TT attended to different courses in Mathematics Education, carried out by the authors themselves and by another colleague involved in the ArAl project. Many lessons were devoted to the problem of the teaching of algebra, with a specific focus on the main new trends in Early Algebra and on the role played by the teacher in the class. The first laboratorial activity in which the TT were involved was focused on a problem that we are going to analyse in detail in the following section. The TT were asked to propose the problem to their students and to carry out its resolution during a whole class discussion. This is the text of the problem:

"A florist sends to a flower grower an email, asking to send him plants of sage and rosemary. However he does not indicate the exact number of plants, but specifies that for every 4 plants of sage he wants 6 plants of rosemary. Let r represents the quantity of plants of rosemary and s the quantity of plants of sage. Represent: (a) the relation between these two quantities; (b) the number of plants of sage through the number of plants of rosemary; (c) the number of plants of rosemary through the number of plants of sage.

Draw, in the Cartesian plane O(r,s), the graph of the relation that expresses the quantity of plants of sage through the quantity of plants of rosemary. Then draw, in the plane O(s,r), the graph of the relation that expresses the quantity of plants of rosemary through the

quantity of plants of sage.
The flower grower delivers 66 plants of sage and tells him that he will send later the plants of rosemary. How many plants of rosemary does he have to deliver?"

This problem can be located among the activities, usually proposed in grade 7, which are aimed at linking the discrete arithmetic of natural numbers to the arithmetic of rational and real numbers.

We have chosen this specific problem because it could be both faced: (1) Adopting an "arithmetical approach" - typical of the Italian school tradition - focused on the application of properties of proportions; (2) Referring to an idea of proportionality as a functional relation, idea that could open the way to the development of algebraic reasoning. The approach that we have suggested TT to adopt in their classes was the second. Our aim was, in fact, to make them become aware that classical problems that, according to the Italian school tradition, are usually faced through the application of "rules" could instead be solved through the study of relations and, therefore, through the activation of an interesting game of interpretations. This approach to the resolution of the problem involves, in fact, the intertwining of different interpretative levels (as we will highlight in the next section) and the activation of different representations of the proportional relation involved.

The objectives of the problem and the main processes to be activated during a class discussion on its resolution were therefore shared with TT with the aim of making them recognise the potential of the task to elicit algebraic reasoning in students.

RESEARCH AIMS AND RESEARCH METHODOLOGY

In this paper we are going to focus on our analysis of the transcripts of the class discussions conducted by TT. Our main aim is to highlight what kind of difficulties they faced when they implemented the problem in their classes, trying to follow our suggestion of adopting an 'algebraic approach' to its resolution (namely an approach focused on the algebraic representation of the relation involved and aimed at the activation of different interpretative processes).

In the following we will analyse the problem introduced in the previous paragraph. This a-priori analysis of the problem has two main aims:

(1) The first aim is to highlight the multifaceted interpretative processes that could be activated in its resolution. Since this analysis was shared with TT, this "hypothetical" resolution represents also the path that they were asked to follow when facing the problem during the whole class discussions with their classes.

(2) The second aim is to identify specific indicators for our analysis of the transcripts of the class discussions conducted by TT. These indicators refer to the games of interpretations and the meta-level reflections that TT should have tried to develop in their interaction with students.

Analysis of the problem and identification of the indicators for the analysis of TT' ways of guiding their classes in its resolution

As we stated above, if we focus on an "algebraic approach" to the resolution of the problem, it could be characterised by a deep twine between aspects related to the use of different representation registers and aspects related to the activation of interpretative processes at different levels.

To carry out the task in the class the teacher should initially guide students in the analysis of the text of the problem and in the identification of the key verbal relation that has to be translated into a mathematical sentence. This requires to consider proportions between couples of numbers that exemplify the relation itself and their classical representation (i.e. 8:12=4:6, 12:18=4:6…), then to translate them into fractional terms (i.e. 8/12=4/6, 12/18=4/6…).

A discussion aimed at a real sharing of the meaning of the letters r and s, introduced in the text of the problem, should precede the formalization of the relation between the number of plants of sage and the number of plants of rosemary. The possible formalizations of this relation should be compared and interpreted through their verbalisation. For instance: if the relation is represented through the proportion $s:r=4:6$, it can be interpreted as "the quantity of plants of sage is to the quantity of plants of rosemary as 4 is to 6"; but it could also be represented through $s:4=r:6$, that can be interpreted as "the fourth part of the quantity of plants of sage is equal to the sixth part of the quantity of plants of rosemary"; etc. This moment, devoted to the interpretation of the formalizations proposed by students, prevents from the uncritical acceptance of erroneous symbolic formalizations such as $4s=6r$, where letters play the role of simple labels.

During this discussion, the teacher should also foster the representation and subsequent verbal interpretation of the previous proportions in terms of equivalence between fractions. This leads to the required representation of the number of plants of sage through the number of plants of rosemary and vice versa. For example, starting from the equality $\frac{s}{r}=\frac{4}{6}$ and the simplified $\frac{s}{r}=\frac{2}{3}$, the class could interpret the latest as "the ratio between s and r is 2/3" and then as "s is 2/3 of r", that can quickly lead to $s=\frac{2}{3}r$, which is the required symbolic representation of the number of plants of sage through the number of plants of rosemary.

After the identification of the two representations - $s=\frac{2}{3}r$ and $r=\frac{3}{2}s$ - of the relations between s and r, another interpretative process should be activated: the particularization of the two formulas through the analysis of specific numerical cases, which could foster students' acquisition of the concept of variable. The couples of values determined through this particularization could be inserted into two tables that better enable to highlight the interrelations between one couple of values and the one that is obtained inverting the values.

After the construction of the two required graphs through the identification of the points corresponding to the couples of values collected in the two tables, the teacher should guide students in noticing that, although the alignment of the points could induce the idea of drawing a continuous line, not all the couples of numbers that are solutions of the two equations are also representatives of the phenomenon that has been modelled. Another aspect to be discussed in this phase is the pertinence of the solution r=99, s=66 according to the florist's request. These observations could be followed by a geometrical examination of the model, discussing the meaning of the two ratios 2/3 and 3/2 and introducing the neutral reference system (x,y) to compare the graphs of the two equations and empirically identify their geometrical relation.

This analysis enabled us to identify four main key-phases that characterise the resolution of the problem and four main groups of indicators, corresponding to the different phases, which are summarised in the following table:

Key-phases in the resolution of the problem	**The teacher guides the students in the:**
Phase 1: *The transition from proportions to the formalization of the proportionality law*	– Identification of the couples of numbers that satisfy the condition required in the problem; – Generalization and corresponding construction, through the introduction of letters, of the proportions representing the relation between the two quantities; – Verbalization of the constructed proportions; – Interpretation of the proportions in fractional terms.
Phase 2: *The twine between syntactical and semantic aspects in the transition from the implicit forms of the relation to its two explicit forms*	– Control of the syntactical transformations that lead to the two explicit formulas; – Identification of the calculation process subtended to each formula; – Verbalization of the meaning expressed by each formula; – Conceptualization of the letters as variables and identification of their different roles (independent vs dependent variable); – Discovery of the predicting power of each formula; – Conceptualization of each formula as an equation and of the couples of numerical values that verify it as solutions of this equation; – Discovery of the direct connection between the two formulas and exploration of the interrelation between their solutions.
Phase 3: *The representation of the two relations on the Cartesian plane.*	– Coordination between symbolic and graphic registers to represent the graphs of the two formulas in the Cartesian planes (r,s) and (s,r); – Re-nominalization of the variables to represent both the formulas in the Cartesian plane (x,y);

	– Interpretation of the graphs in relation to the problem and discovery of their predicting power.
Phase 4: *Control of the adherence of the mathematical model to the specific problem-situation.*	– Comparison between the domain and codomain of each relation and the domain and codomain of the corresponding restrictions that model the problem-situation; – Reflection on the acceptability of certain couples of values (e.g. s=66 and r=99) as solutions of the problem.

Table 1: Indicators for the analysis of the transcripts of the discussions conducted by TT in their classes

As we stated above, this a-priori analysis was shared with TT before they proposed the problem to their classes. TT were asked to audio-record the discussions they conducted with their students and to reflect on these discussions referring to three different perspectives: the mathematical content at play, the role played by the teacher, the students' approaches to the problem and reactions to the teacher's interventions. After they performed this task, they sent us both the transcripts of their class discussions and the corresponding reflections.

We therefore analysed both the transcripts themselves and the TT' reflections and the results of our analysis were discussed with them during a further lesson. In this paper, because of space limitations, we are focusing only on our analysis of the transcripts of TT' discussions, aimed at highlighting the difficulties they faced when they implemented the problem in their classes trying to follow our suggestions. The transcripts' analysis, whose main results will be presented in the next section, was performed referring to the four key-phases and the corresponding indicators that we have identified thanks to the a-priori analysis of the problem.

DATA ANALYSIS: IDENTIFICATION OF SOME PROBLEMATICAL ASPECTS IN THE DISCUSSIONS CONDUCTED BY TT

Through our analysis of TT' discussions we have identified specific problematic aspects, connected to their incapability of guiding students within the games of interpretation and reflection necessary to make them develop those competencies and awareness that are objectives of an "algebraic approach" to this kind of activities.

As regards the first phase in the resolution of the problem (*the transition from proportions to the formalization of the proportionality law*), some TT were not able to support students in abandoning the quantitative/numerical level. They adopted a procedural approach, making students only formulate the numerical proportions necessary to determine the couples of values that satisfy the relation and directly construct the graphs from the table of values. The modeling process was therefore inhibited.

In some classes students were able to formulate different proportional laws starting from the numerical examples they constructed, but the TT accepted all these laws without asking students to interpret them through their verbalization. Many TT declared, during the class discussions, that “all the constructed proportions are the same proportion”, instead of highlighting that, although their equivalence, the relations expressed by these proportions are different. This erroneous conception, probably induced by some textbooks, could be overcome only through the activation of interpretative processes.

As regards the second phase in the resolution of the problem (*the twine between syntactical and semantic aspects in the transition from the implicit forms of the relation to its two explicit forms*), we have highlighted a really alarming phenomenon. Most of the TT did not support students in controlling and discussing the meaning of the two formulas derived from the proportional laws. In fact, very few of them made students analyse the formulas through their particularization to introduce the role played by the letters as variables; almost none of them made students verbalize the formulas they had determined.

Another problematical aspect connected to this phase is that different TT often used terms such as “find s/r; search for s/r; calculate s/r” - instead of “make s/r explicit” or “express s/r through r/s” - to invite students to construct the two explicit formulas. Also the use of this procedural language inhibits the conceptualization of variable. Finally, most of the TT guided students in the determination of the two explicit formulas starting from the constructed proportions, but they did not highlight how they could be obtained from each other, nor foster a reflection on their relationships.

As regards the *third phase* in the resolution process (*the representation of the two relations on the Cartesian plane*), an aspect that we have highlighted is that some TT let that students construct the graphs determining the couples of values that are solutions of the two explicit equations through proportions, without stressing on the predicting power of the two formulas. In our view, this behaviour testifies that they conceive the modeling phase as something unnecessary and that they do not interpret the two formulas as the representation of all the possible pairs of numbers having the same ratio. Moreover, the fact that students do not refer to the formulas in constructing the graph is an evidence of a lack in their control of the meaning that the formulas convey.

Another widespread problem related to the TT’ capability of coordinating the verbal, the algebraic and the graphic registers is that, although many TT examined with students the geometrical properties of the graphs, almost none of them made students interpret the graphs, highlighting their predicting power. Moreover they accepted continuous lines or semi-lines as graphs of the relations, without developing a reflection on the fact that only some points of these lines are representatives of the phenomenon that has been modelled.

This last aspect is also connected to the problematic ones that we have noticed in analysing the TT’ discussions according to the indicators concerning the *fourth phase* of the resolution process (*control of the adherence of the mathematical model to the* *specific problem-situation*). Few TT, in fact, made students reflect on the domain and codomain of the relations that model the problem-situation and on the acceptability of certain couples of values as solutions of the problem (during most of the discussions the value 66 as a possible number of plants of sage was uncritically accepted).

CONCLUSIONS

Through the analysis of the transcripts of TT' discussions, we highlighted that, although they had shared with us the aims of the activity and the a-priori analysis of the problem, many of them were unable to activate the necessary interpretative processes and meta-level reflections and therefore to exploit the potential of this task to elicit algebraic reasoning in students.

The evident blocks in the games of interpretation that should have been activated during the class discussions testify corresponding profound gaps in the TT' knowledge. These gaps, in fact, prevent them from becoming aware of what it is important to stress when facing this kind of activities with students. If teachers do not overcome these difficulties, they will not be effective models for their students, making them develop erroneous attitudes. This is a crucial problem in the didactic of algebra, because of the plurality of levels of interpretation that a teacher is required to effectively control.

These results suggest that research should better scrutinize teachers and students' difficulties in coordinating the different levels of interpretation often involved in the algebraic activity, in order to identify possible strategies to overcome them. At the same time, as it was also stressed in the discussion within the TWG3, a reflection should be developed on how teacher education programs must be engineered according to these results with the aim of enabling teachers to become effective activators of interpretative processes.

REFERENCES

Arcavi, A. (1994). Symbol sense: informal sense-making in formal mathematics. *For the Learning of Mathematics*, 14(3), 24–35.

Arzarello, F., Bazzini, L., & Chiappini, G. (2001). A model for analyzing algebraic thinking. In R. Sutherland, T. Rojano, A. Bell & R. Lins (Eds.), *Perspectives on School Algebra* (pp. 61-81). Dordrecht: Kluwer.

Bell, A. (1996). Problem solving approaches to algebra: two aspects. In N. Bednarz, C. Kieran & L. Lee (Eds.), *Approaches to algebra. Perspectives for research and teaching* (pp.167-187). Dordrecht: Kluwer.

Blanton, M.L., & Kaput, J. (2005). Characterizing a Classroom Practice That Promotes Algebraic Reasoning. *Journal for Research in Mathematics Education*, 36(5), 412-446.

Blanton, M.L., & Kaput, J. (2011). Functional Thinking as a Route Into Algebra in the Elementary Grades. In J. Cai & E. Knuth (Eds.), *Early* *algebraization: A global dialogue from multiple perspectives* (pp. 5-23). Berlin Heidelberg: Springer.

Clement, J. (1982). Algebra word problem solutions: thought processes underlying a common misconception. *Journal for Research in Mathematics Education*, 13(1), 16-30.

Cusi, A., Malara, N. A., & Navarra, G. (2011). Early Algebra: Theoretical Issues and Educational Strategies for Bringing the Teachers to Promote a Linguistic and Metacognitive approach to it. In J. Cai, & E. J. Knuth (Eds.), *Early Algebraization: Cognitive, Curricular, and Instructional Perspectives* (pp. 483-510). Berlin Heidelberg: Springer.

Duval, R. (2006). A cognitive analysis of problems of comprehension in a learning of mathematics. *Educational Studies in Mathematics*, 61, 103–131.

Kieran, C. (1992). The learning and teaching of school algebra. In D. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 390-419). New York, NY: Macmillan Publishing Company.

Kieran, C. (2004). Algebraic thinking in the early grades: What is it? *The Mathematics Educator*, 8(1), 139-151.

Magiera, M.T., van den Kieboom, L.A., & Moyer, J.C. (2013). An exploratory study of pre-service middle school teachers' knowledge of algebraic thinking. *Educational Studies in Mathematics*, 84, 93-113.

McCrory, R., Floden, R., Ferrini-Mundy, J., Reckase, M. D., & Senk, S.L. (2012). Knowledge of Algebra for Teaching: A Framework of knowledge and practices. *Journal for Research in Mathematics Education*, 43(5), 584-615.

Strand, K., & Mills, B. (2014). Mathematical content knowledge for teaching elementary mathematics: a focus on algebra. *The Mathematics Enthusiast*, 11(2), 385–432.

Ursini, S. (1990). Generalization processes in elementary algebra: interpretation and symbolization. In G. Booker, P. Cobb, & T.N. Mendicuti (Eds.), *Proceedings of the 14th Conference of the International Group for the Psychology of Mathematics Education*: Vol. 2 (pp. 149-156). Mexico: PME.