Generalization questions at early stages: the importance of the theory of mathematics education for teachers and pupils

Nicolina A. Malara, Giancarlo Navarra

We discuss the introduction of early algebra in primary school, stressing the role of the theory of mathematics education for a reconstruction of the teachers' knowledge, awareness, beliefs and behaviors and to promote the teaching of arithmetic in a pre-algebraic perspective. We present some excerpts of classroom episodes, from experimentations carried out in collaboration with teachers, which testify the influence of new ways of teaching on the pupils' performance.

1. Introduction

The identification of relationships and their algebraic modeling are central activities in the curricula of many countries and pervasively present in the PISA tests. Nevertheless, in many countries they are hardly dealt with in everyday teaching. The main reasons for such situation are manifold: the persistence of teaching models aiming only at computational techniques, the prevalence of activities of procedure reproduction in most textbooks, the common attitude of fostering knowledge of mathematical facts rather than looking for their justification, the weak fallout of consolidated research results in the academic training of future teachers, the inadequate investment of institutions in long term programs for in-service teachers.

Didactic research has highlighted the importance of emphasizing a relational view of mathematics right from the early years of schooling, particularly in arithmetic, so as to promote the pupil's ability to see the 'general in the particular' and to make the shift to the algebraic language as a 'tool for thinking' meaningful and less traumatic (Kieran 1992, Bell 1996, Mason 1996). In the 90's, in order to avoid obstacles due to the standard procedural teaching of arithmetic, pre-algebra was introduced as an approach dedicated to the encoding of processes in problem solving, the objectification of calculation sequences so as to highlight patterns and structural analogies (Linchevski 1995). Early algebra became a specific subject area regarding the interaction between arithmetic and algebra, and it is now present as such in the curricula of various countries (Cai et al. 2005). It combines purely disciplinary aspects with methodological aspects, allowing time for the interaction between pupils, for argumentation, for reflection on the mathematical meanings which arise during classroom activities. Its linguistic dimension acquires a dual role: a tool for the construction of mathematical objects but also to conceptualize knowledge.

Many experimental studies of classroom implementation of early algebra have simultaneously tackled the issue of teacher training (see, for example, in Cick et al. 2001 the reports by Carpenter et A., Kaput & Blanton, Daugherty). Later studies have highlighted the importance of the teacher's 'attitude' in order to promote the right point of view in the pupils – Radford, for instance, speaks of 'domestication of the eye' (Radford 2010, p. 4).

Moving from this background, our studies are characterized not only by the construction of a theoretical framework made available for teachers and supported by a specific glossary, but also by the strong interaction between innovating classroom activities and training teachers *in* the classroom (Malara & Navarra 2003, Cusi et al. 2010). Such interaction aims at the 'critical analysis of classroom processes, particularly of the teacher's action'. The methodology adopted, explained in detail further on, has produced a framework of theoretical constructs which is crucial for the communication among teachers, for our own communication with them, for classroom activities or for teacher-pupils interaction. Such constructs give a concrete example of the role that theory has in mathematics education when it is properly mediated, i.e. as a key to designing a teacher role in line with actual didactical needs, specifically in fostering in the pupils a fruitful attitude towards generalization besides an aware and effective learning.

We report here some significant elements of our theoretical framework and methodology then conclude with meaningful examples regarding the beliefs that arise in the pupils, which we then discuss in the workshop.

2. Early algebra

Early algebra derives from the assumption that *the main obstacles in the learning of algebra arise, even in unexpected ways in arithmetic contexts, when pupils are still very young.* These obstacles *may affect the development of mathematical thought* by allowing only a weak conceptual control on the *meanings* of *objects* and algebraic *processes*.

Our research operates in the setting of *early algebra,* within a socio-constructive teaching model: the activities develop when pupils face purposely designed problem situations, through which the teacher fosters the rising of mathematical concepts and properties.

The most important aspects of this approach are:

• *the anticipation of pre-algebraic activities of generational kind* at the beginning of primary school or even early - in kindergarten;

• *the social construction of knowledge*, negotiated on the basis of the cultural instruments held in common at that particular moment by the pupils and the teacher;

• *the centrality of natural language* as main teaching mediator for the construction of the semantic and syntactic aspects of the algebraic language;

• *the identification and highlighting of the algebraic thinking enclosed in concepts and representations of arithmetic*, e.g. guiding pupils towards the interpretation and description of a sentence like $4 \times 2 + 1 = 9$ from a procedural reading ('I multiply 4 by 2, add 1 and obtain 9') to a relational reading ('The sum of the product of 4 with 2 and 1 is equal to 9 '). The focus is shifted from the objects with which one operates towards *the relations between them,* by analysing the structure of the sentence.

The teacher plays a central role in these processes. To do that, he/she is guided into a profound reinterpretation of his/her knowledge and beliefs about arithmetic and algebra, so that they can capture the mutual relations between the disciplines and the embryos of algebraic thought hiding in the concepts and representations of arithmetic.

3. The role of the ArAl glossary in teacher training

In the ArAl project, the image of early algebra is expressed through a set of key words and concepts that refer to arithmetic and algebra, but it defines its areas evolving from both disciplines towards a different and original identity. We can consider it a meta-discipline, concerning not much the objects, processes and properties of arithmetic and algebra, but rather the genesis of a unifying language between the two, i.e. a meta-language. In order to control the meta-disciplinary knowledge of early algebra, the teacher acquires the meaning of its basic vocabulary.

All the keywords of the Glossary have been attained through a slow process of reflection on the basic concepts of the two disciplines, which led to a consolidation of the specific knowledge of early algebra. They must therefore be understood as traces of the construction of knowledge in a process in continuous evolution, capable of giving a representation of itself.

These foregoing considerations show the centrality of the Glossary (available on the website www.progettoaral.wordpress.com), currently consisting of nearly one hundred and fifty interconnected lexemes, referring to five different areas: general, linguistics, mathematics, socio-educational, psychological. This reference system enables the teacher to approach a *linguistic* conception of algebra in which pupils build together a convincing control of its meanings.

4. Some basic concepts

We present here some of these terms seen as key concepts in our approach to early algebra: algebraic babbling; representing vs. solving; syntax / semantics; translations between languages; canonical form / non-canonical form of the number; the '=' sign.

4.1. Algebraic babbling

The control of the syntactical aspects of a new language is obtained through its semantic control. On acquiring a natural language, the child gradually realizes its meanings and the rules that support them, which gradually develop up to school age, when he learns to read and reflect on the *structural* aspects of the language. Similarly, it is believed that the mental models of algebraic thinking should be organized right from the early years of primary school, constructing algebraic thinking, in a tight intermingling with arithmetic, by starting from its *meanings*. It is therefore necessary to build an environment promoting the autonomous development of formal encodings of sentences in the natural language and their collective comparison. This allows the experimental acquisition of the new language, whose rules gradually ripen within a didactic contract which *tolerates* initial, syntactically shaky moments. We call this process of construction/interpretation/refinement of the 'rough' writings *algebraic babbling*.

For example, in a third primary grade, the teacher accepted for the phrase 'the double of a unknown number' the following translation: 'd. of u.n.'. A first 'cleaning' of this sentence is performed by the class as soon as they discover that an unknown number, in the algebraic language, is indicated by a single letter, so the phrase is changed into 'n of d'. The reflection on the meaning of 'twice a number' leads to the concept of 'number multiplied by 2' and so the translation becomes 'n × 2'. In this process, the teacher helps to interpret the writings, orchestrates the discussion of the meanings and facilitates the spotting out of the most appropriate translations.

4.2. Algebra as a language: representing vs. solving

Encouraged by the traditional teaching of arithmetic, it is a widespread belief among pupils that the solution of a problem coincides with the detection of its result. This implies that their attention is focused on *operations*. They should instead be slowly oriented from the cognitive towards the *meta-cognitive* level, at which the solver *interprets the structure of the problem and represents it through the language of mathematics*.

The development of arithmetic thought, characterized by operations on known numbers, may result in the formation of hardly extinguishable stereotypes, because of which the pupils get caged in the obsessive search for a numerical result, thus hindering the exploration of different, much more effective mental paths, stimulating for the formation of an embryonic algebraic thinking.

The following example highlights the difference between the tasks 'Solve' and 'Represent':

Eva has 13 Euros. She receives 9 more Euros and spends 6 Euros.

A. 'classical' task in arithmetic perspective: How many Euros are left to Eva?

B. Task in algebraic perspective: Represent the situation in the mathematical language so that others can find out how many Euros are left to Eva.

Task A emphasizes the search for the product (16), whereas B concentrates on the process (13 +9-6), i.e. the representation of the relationships between the elements in play. This difference is connected to one of the most important aspects of the epistemological gap between arithmetic and algebra: while arithmetic implies an immediate search for solution, algebra delays it and begins with a formal transposition of the problem situation from the domain of natural language to a specific system of representation.

4.3. Respecting the rules: syntax and semantics

Closely related to the act of representation is the issue of *respecting the rules* in the use of a language, even more necessary when using a formalized language. In everyday life, respecting the rules is favored by family, society and - from a certain point onwards - by school, which stimulates a reflection on the structural aspects of language. On teaching of mathematics, rules are generally 'delivered' to pupils, thus losing their social value of support to the understanding and sharing of a language as a communication tool. You must then bring the pupils to understand that they are acquiring a new language which, like all languages processed by man, has a grammar and a syntax system (which are a set of conventions allowing to construct sentences correctly), but also semantics (which allows to interpret the symbols within syntactically correct sequences and determine whether the obtained expressions are true or false). Despite the fact that the pupil internalizes from birth the fact that compliance to the rules allows communication, it is highly unlikely that he will transfer this peculiarity to the mathematical language. In order to overcome this step, we ask pupils to exchange messages in arithmetic-algebraic language with *Brioshi, an algebraic pen pal* - a fictitious Japanese pupil who speaks only in his mother tongue, whose age ranges according to the age of his dialogue partners. This trick works as a powerful didactical mediator to highlight the importance of respecting the rules while using this language.

4.4. Translations between languages

In this perspective, *translating* from the natural language to the mathematical one (and vice-versa) is a good occasion to develop reflections on the language of mathematics. It means *interpreting* and *representing* a problematic situation by means of a formalized language or, on the contrary, recognizing the situation that it describes in a symbolic writing. This activates the pupils' control of the two registers of expression on one hand, developing on the other hand the meta-cognitive ability to understand how syntactic transformations of formal expressions condense thought processes that would hardly be achieved by using natural language. This questions contains the very delicate issue of the different meanings associated to the '=' sign, which often are not spelled out.

4.5. Canonical/non canonical form of the number

Faced with the question: 'Is $[3 \times (11 + 7)\ 9]^2$ a number?' Italian pupils, but also in-training teachers, usually answer saying: "No, these are operations", "It's an expression, ""They are calculations." To promote reflection on this aspect we generally write on the blackboard some information about one of the pupils in the class. We create lists such as: 1) Marika; 2) Laura's daughter; 3) Matthew's daughter; 4) Christian's sister; 5) Renato's granddaughter; 6) owner of the dog called Floppy; 7) living in Such and Such Street number 24 ...

The pupils understand that all these expressions are different *ways to name* the classmate: Marika has her own name and all other descriptions (*representations*) are based on her relationships with other individuals, which broaden our knowledge of her by adding information her name does not convey. The teacher then explains that the situation is similar with numbers: each number can be represented in different ways, through any expression equivalent to it. One (e.g. 12) is its name, the so called *canonical form*, all other ways of naming it (3 × 4, (2 + 2) × 3, 36/3, 10 +2, 3 × 2 x 2, ...) are *non canonical forms*, and each of them will make sense in relation to the context and the underlying process. This experience allows older pupils to conclude that [3 × (11 +7) 9] 2 is *one of the many non-canonical forms* of the number 36.

Knowing how to recognize and interpret these forms creates the semantic basis for the acceptance and understanding of algebraic writings such as -4p, ab, x^2y, k/3. The complex process that accompanies the construction of these skills should be developed throughout the early years of school. The concept of a canonical/non-canonical form has for pupils (and teachers) implications that are essential to reflect on the possible meanings attributed to the sign of equality.

4.6. The sing '='

On reading (for example) 6 +11-2 = 15, teachers and pupils often 'see' the operations to the left of the sign and a result to the right of it. The prevalent idea is: 'I sum 6 and 11, then subtract 2 and obtain 15'. In this perspective, the 'equal' sign expresses the meaning of *directional operator* and has a mainly space-time connotation: it prepares the conclusion of a story (calculations) which should be read from left to right up to its conclusion (the result).

When shifting to algebra, however, this sign acquires a different meaning. In a writing such as a 2a-6=2 (a-3) it assumes a *relational* meaning, since it indicates the equivalence between two representations of the same quantity. Therefore the pupils must learn to move in a conceptual universe in which it is necessary to go beyond the familiar space-time connotation. If the rooted concept implies that 'the number after the equal sign is the result', a writing such as 26=x-7 is likely to mean very little to them, even though they might know how to solve the equation.

In primary school, for instance, the task 'Write 14 plus 23' often gets the reaction '14+23 ='. The 'equal' sign is therefore considered a 'necessary' *signal of conclusion*, expressing the belief that a conclusion is sooner or later required by the teacher. '14 +23' is seen as an event that is *waiting to be fulfilled*. The basic operation attitude prevails as a consequence of a didactic centered on calculations. The absence of the '=' sign is seen as a *missing conclusion* to a transaction, as if the writing 14 +23 (without the 'equal' sign) were 'incomplete'. The pupils suffer here from lacking or poor control over meanings.

When faced with questions such as the representations of the number and the meaning of the equal sign, Italian teachers are often unarmed at epistemological level. This confirms the fundamental role of their education and training.

5. Teachers and classroom discussion

The scenario that we have so far described requires a change of perspective in the teachers, who now learn a new way of managing the *socio-cognitive* processes by comparing their epistemology with the frameworks we suggest. On gradually developing the results of such comparison, they turn them into a steady cultural heritage. This process aims at their recognizing in themselves (and in the pupils) some new behavior, which wouldn't have arisen previously. The key aim of the training process we propose is, therefore, to lead teachers to *new awareness* by pointing out *which aspects* they should focus on. Besides, we help them understand *how* to intervene in the classroom.

It is therefore clear for teachers that since the construction of knowledge takes place through the promotion of social dynamics that encourage discussion and verbalization, mathematical *discussion* obtains a central role in this process. So they learn to guide it consciously, by activating very complex skills. Fundamental in this respect are the activities of critical reflection on the classroom processes, which we describe in the following paragraph.

6. The method of 'pluri-commented' transcriptions

Our methodology with teachers is based on the assumption that the theoretical study of research results, the observation and the critical reflection of mathematical discussions are essential in order to make teachers acknowledge any gap between their declared beliefs and the implicit ones (revealed by their actions or even between theoretical assumptions they share with the researchers and the practices they actually carried out in the classroom). It is based on the critical analysis of the transcripts, made by teachers themselves, of audio-recorded classroom activities (called 'diaries'). The diaries are sent by email and then commented on by a variable number of actors - the class teacher, his/her e-tutors, other teachers, teacher-researchers and university researchers. The commentaries concentrate on the conduction mode, on the content, on linguistic aspects and the social dynamics. This methodology gives the teachers the opportunity to retrace analytically the activity carried out, besides critically reviewing their own work in the light of the comments arisen. Thanks to these practices of sharing and joint reflection, teachers gradually acquire awareness of the dynamics and variables involved in the collective construction of mathematics, hence sharpening their ability to capture pupils' insights and potential, while learning to have more control over their way of being in the class.

7. The Workshop

During the Workshop, the participants discuss some micro-episodes taken from our experimentations (grades 1-8) where teachers worked in the context of early algebra. These episodes show how concepts are developed in the pupils, thus offer opportunities for thoughtful reflection on a key question: *when and how does the curtain on algebra begin to open?* For reasons of space, we limit ourselves here to a few examples:

• Piero (8 years) observes that "It is correct to say that 5 plus 6 makes 11, but you cannot say that 11 'makes' 5 plus 6, so it is better to say that 5 plus 6 'is equal' to 11, because in this case the other way round is also true." Piero is discussing *the relational meaning of the equal sign*.

• Miriam (9 years) represented the total number of sweets contained in six bags (each of which contains four chocolates and three candies) as follows: (3 +4) × 6. She reviews Alessandro's writing

(7 × 6) by saying "What I wrote is more transparent, Alessandro's writing is opaque. Opaque means that it is not very clear, whereas transparent means clear, that you understand." Miriam reflects on *how the non-canonical form of a number helps to illustrate the structure of a problematic situation*.
• Lorenzo (10 years old) reads what he has written for the task 'Translate the sentence 3 × b × h into natural language': "I multiply 3 by an unknown number and then I multiply it for another unknown number." After hearing Rita's proposal, "The triple of the product of two numbers that you don't know," he observes "Rita explained what 3 × b × h is, whereas I have told what you do." Lorenzo captures *the dichotomy process-product*.
• Diana (11 years old) has the task of representing in mathematical language the sentence "Twice the sum of 5 and its next number." When the pupils' proposals are displayed on the whiteboard, she justifies her writing: "Philip wrote 2 × (5 +6), and it is right. But I have written 2 × (5 +5 +1) because this way it is clear that the number next to 5 is a larger unit." Diana is emphasizing the *relational aspects of the number*.
• Thomas (12 years) has represented the relationship between two variables this way: a = b +1 × 4 and he explains: "The number of the oranges (a) is four times the number of the apples (b) plus 1" Katia replies "It's not right: that would mean that the number of oranges is the number of apples plus 4. You have to put the brackets: a = (b +1)× 4". Thomas and Katia are discussing *the translation between natural and algebraic language* and the *semantic and syntactic aspects* of mathematical writings.
Through these excerpts we will discuss about: (a) the effectiveness of some theoretical constructs in the development of the classroom discussions, allowing the pupils' shift of attention towards an algebraic way of interpreting arithmetical questions; (b) the positive effects of our didactical proposals and methodology of work with the teachers, which foster productive attitudes in the pupils to forward the generalization and the translation of verbal relationships to algebraic sentences.

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