The Analysis of Classroom-Based Processes as a Key Task in Teacher Training for the Approach to Early Algebra

Nicolina A. Malara and Giancarlo Navarra

We synthesise here a theoretical framework outlined for the renewal of the teaching of algebra, giving a prominent role to a linguistic approach to the discipline starting from primary school. We tackle the issue of how teachers can be led to acquire conceptions and models of behaviour suitable to foster such a view of algebra among pupils and, in particular, to develop in them modelling, interpretation and thought production skills. For this we propose a model of task addressed to teachers and devoted to the analysis of activities and pupils' productions, as well as on the design of classroom discussions. The task is aimed at favouring the development of predictive and interpretative thinking by teachers as to pupils' behaviour. The structure, made of enchained scenes, induces teachers to question and review their convictions about their own role in the classroom and to become aware of the dynamic processes involved in the social construction of knowledge.

1 Introduction

Everyday teaching and learning practice is very often characterised by mere transmission of mathematical facts and application of rules, thus neglecting the conquest of the underlying sense. It is extremely rare to find a *teaching practice based on the exploration of problems*, in which focus is shifted *from results to processes*. Such a shift in perspective would give the opportunity to build in pupils – and hence to disseminate in the social context – an image of mathematics more in accordance with its real nature: that of a discipline born *from and for the study of problems*.

Current research points to the *socio-constructivist* approach as the most suitable to raise pupils' interest in mathematics and foster in them a *meaningful* conception of the discipline. This approach is based on a theory according to which the mind constructs schemes through which knowledge develops by selection, organisation and continuous *re-structuring* of facts and relationships, drawing on the richness of sensations and stimuli that come from the environment and from social interactions.

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This has notable implications for teaching and learning: the teachers' role acquires different and more complex features. The teacher cannot be a simple knowledge conveyor any longer: he/she rather becomes a person in charge of creating an environment that enables pupils to construct their own mathematical knowledge.

It is significant that, in recent years, research studies have been dedicated more and more to the teacher, as underlined at the recent ICME-10 congress (Copenhagen, 2004).[1] In particular, a model of teacher was outlined as a *reflexive and critical* decision maker (Jaworski, 1998, 2003; Malara & Zan, 2002; Mason 1998, 2002; Peter-Koop, 2001; Ponte, 2004; Schoenfeld 1998), pedagogically sensitive,[2] able to observe him/herself analytically in the moment he/she acts and to consciously make decisions in real time.

Teacher training according to such a model is necessary to obtain a socio-constructivist teaching practice based on the mastering of meaning as a prerequisite for the growth of the pupils' competencies. The challenge is thus in leading teachers to reflect on both the discipline and the model of teacher they have and apply. Training must be, therefore, an important time for of reflection about central questions such as: *What mathematics do we need to teach? How can we do this? What should I refine, modify or reconstitute in myself in order to fit in the current trends of teaching?*

In this perspective, on one hand teachers need to be offered chances, through both individual study and suitable experimental activities, to revise their knowledge and beliefs about the discipline and its teaching, in order to overcome possible stereotypes and misconceptions. On the other hand, they need to become aware that their main task is to make students able to give *sense and substance* to their experience and construct new knowledge by exploring situations and making links with familiar concepts.

The actual attainment of these goals is strictly related to the type of mathematical content at stake, as well as to teaching traditions and educational policies in the different countries. This is extremely complex in the case of classical thematic areas, such as arithmetic and algebra, which suffer from their antiquity, and the teaching of which is affected by the way they historically developed.

Teacher training referring to *early algebra* currently is of great research interest: this disciplinary area became increasingly important in the last decade, as an answer to issues related to the teaching and learning of algebra. The work we present here concerns this content area and deals with methods and educational processes for

[1] In her plenary lecture, Anna Sfard pointed out the features of research in mathematics education in the last 50 years: from the *'programs' era'* in years 1960s–1970s there was a shift to the *'students' era'* in years 1980s–1990s, to get on the way of the *'teachers' era'* at the beginning of years 2000. In brief: 'good practices' characterising the *'programs' era'*, or knowledge of students' cognitive styles and related changes, typical of the *'students' era'* are necessary but not sufficient for a renewal of teaching if teachers are not given opportunities to refine their mathematical and educational knowledge, to discuss and reflect upon their individual beliefs.

[2] With this term we mean what Jaworski, (2003) calls 'sensitivity to the students' and concerns: how the teacher knows pupils, his/her attention to their needs; his/her interaction with individuals and ability of guiding group interactions.

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teachers, aiming at a renewal of the teaching of arithmetic and algebra in a socio-constructive perspective, developed in co-operation of researchers and teachers.

2 Why Early Algebra

One of the most heartfelt problems for secondary school teachers concerns the difficulties that students have in their approach to algebra. The main reasons for these difficulties essentially lie in the heavy loss of meaning felt by students about the objects of study.

Since the 1980s, research has pointed to a way to modify this situation, underlying the need to promote, since primary school, a pre-algebraic teaching of arithmetic. This type of teaching would cast toward the observation of numerical regularities, the recognition of analogies, generalisation and an early use of letters to represent observed facts (Davis 1985; Linchevski 1995).

Starting from the second half of the 1990s, many theoretical and experimental studies were carried out on these aspects, mainly addressing 11–13-year-olds. Some of these studies stand out due to a theorisation of socio-constructive models of conceptual development; they emphasise the influence of the class environment on learning and promote the use of physical instruments as means of semeiotic mediation, in the frame of a view of algebra as a language (Da Rocha Rocha Falcão,1995; Meira, 1990, 1996; Radford, 2000; Radford & Grenier, 1996).

Since 2000, broad studies concerning teaching experiments, carried out at primary school level in relation to algebraic contents, appear: resolution of simple equations through problems with the introduction of unknowns; study of relations, functions and sequences; introduction to proof (see for instance Carraher, Brizuela, & Schliemann 2000, or Carpenter & Franke, 2001). Some of these studies also concern the setting up of projects aimed at training primary school teachers on these issues (Blanton & Kaput, 2001; Brown & Coles, 1999; Dougherty, 2001; Menzel, 2001). Our *ArAl Project, paths in arithmetic to favour pre-algebraic thinking* locates within this frame (Malara & Navarra, 2003), and is a project that merges teacher training and innovation in the classroom.

3 Our Hypotheses and Basic Theoretical Elements in the Approach to Early Algebra

Algebra is usually introduced as a study of algebraic forms, privileging syntactic aspects, as if formal manipulation preceded meaning understanding. As a consequence, algebraic language comes to lose some of its essential features: that of being a suitable language for describing reality, by coding knowledge or making hypotheses about phenomena; that of being a powerful reasoning and predicting instrument, that enables the individual to derive new pieces of knowledge about phenomena, by means of transformations allowed by arithmetic-algebraic formalism.

Our fundamental hypothesis is that one cannot focus exclusively on syntax and leave semantics aside, but rather that there is a need to start from the latter, taking an approach to the teaching of algebraic language in analogy with the learning modalities of natural language. We make use of the babbling metaphor to explain this perspective.

3.1 Algebraic Babbling

In their early years children learn a language gradually, appropriating its terms in relation to the meanings they associate with them, and develop rules gradually, through imitation and adjustments, up to the school age, when they will learn to read and reflect on grammatical and syntactic aspects of language.

Our hypothesis is that the mental models that characterise algebraic thinking must be constructed since the early years of primary school, when children approach arithmetical thinking: this is the time to teach them to think about arithmetic algebraically. In other words, algebraic thinking should be built up in children progressively, in a rigorous interrelation with arithmetic, starting from the meanings of the latter. Meanwhile, one should pursue the construction of an environment which can informally stimulate the autonomous processing of what we call algebraic babbling, i.e. the experimental and continuously redefined mastering of a new language, in which the rules can gradually find their place within a teaching situation which is tolerant of initial, syntactically "shaky" moments. In this process, a crucial point is represented by understanding the difference between the concepts of representing and solving.

3.2 Representing and Solving: Process and Product

A very common pupils' belief is that the solution to a verbal problem is essentially the statement of a result. This naturally implies that attention is focused on what produces that result: operations. Let us consider the following problem that poses a classical question: *There are 13 crows perched on a branch; another 9 crows arrive at the tree while 6 of the previous ones fly away. How many crows are now on the tree?*

Now let us modify the question: *'Represent in mathematical language the situation so that we can find the total number of crows'*. Where is the difference between the two formulations?

In the first case, the focus is on the identification of the *product* (16), whereas the second concentrates on the identification of the *process (13 + 9 – 6)*, that is, the representation of the relationships among the elements in play.

This difference is linked with one of the most important aspects of the *epistemological gap* between arithmetic and algebra: whilst arithmetic requires an *immediate* search of a solution, on the contrary algebra *postpones* the search of a solution and begins with a formal trans-positioning from the dominion of a natural language to a specific system of *representation*. If guided to overcome the worry of the result, each pupil reaches an upper level of thinking, substituting the calculations with the observation of him/herself reasoning. He/she passes to a meta-cognitive level, interpreting the *structure* of the problem.

3.3 Canonical or Non-Canonical Representation of a Natural Number

Among the infinite representations of a number, the *canonical* one is obviously the most popular. Thinking of a number means for anyone thinking of the *cardinality it represents*. But the canonical representation is *meaning-wise opaque*, as *it says little about itself* to the pupil. For instance: the writing '12' suggests a certain 'number of things', or at most the idea of 'evenness'. Other representations – always suiting pupils' age – may broaden the field of information about the number itself: '3×4' points out that it is a multiple of both 3 and 4; '$2^2 \times 3$', that it is also a multiple of 2; '$2 \times 2 \times 3$' leads to '2×6' and therefore to the multiple of 6; 36/3 or 60/5 that it is sub-multiple of other numbers and so forth.

We can say that each possible connotation of a number adds information to get to a deeper knowledge of it, as it happens with people: there are the first and family name, opaque if compared to other more expressive connotations of the subject, for instance with reference to other individuals he or she is linked to by social or family relationships (father of ..., teacher of ..., brother of somebody's husband ...). It is extremely important that pupils learn to see as appropriate the canonical representation of a number as well as any other arithmetical expression of which such number is the result (non-canonical representation of the number). In the case of twelve, appropriate and acceptable representations besides '12' are also '9 + 3' or '$2^2 \times 3$'. This is done not only to favour acceptance and understanding of algebraic written expressions like 'a + b' or 'x^2y', but mainly to facilitate the identification of numerical relationships and their representation in general terms.

In relation to this, there is a significant conceptual challenge, e.g.: in $15'(4+2) = 90$ a pupil, operating a reading left/right 'sees' $15'(4+2)$ as an 'operation' and '90' as its 'result'. But he/she has to be educated to 'see' the sentence as an equality between two representations of the same number. The following section is devoted to this aspect.

3.4 The Equal Sign

In primary-school teaching of arithmetic, the equal sign essentially takes up the meaning of *directional operator*: 4 + 6 = 10 means to a pupil 'I add 4 and 6 and

I find 10'. This is a dominant conception in the first seven or eight school years during which the equal sign is mainly characterised by a *space-time* connotation: it marks the steps of an operative simplification or reduction path (operations are carried out sequentially) which must be read from left to right up to its end (i.e. the reaching of the result). Later, when the pupil meets algebra, the equal sign suddenly takes up a totally different, *relational* meaning. In a written expression like $(a + 1)^2 = a^2 + 2a + 1$ it carries the idea of a symmetry between the expressions: it points to the fact that they represent the same number, whatever the value given to a, and the two expressions are said to be equal (in fact, they identify because they are equivalent with respect to the relation 'representing the same number, as a varies'). Again, in the writing '8 + x = 2x – 5' the equal sign points to the (*still unverified) hypothesis about the equivalence of the two writings for some value of the variable x*. Although nobody told him about this broader meaning, the student must now move into a completely different conceptual universe, where it is necessary to go *beyond* the familiar space-time connotation. But if the student thinks that 'the number after the equal sign is the *result*' he or she will be lost and will probably attach little meaning to a writing like '11 = n', although he/she might be able to solve the linear equation leading to it.

This reasoning shows an evident correlation to *linguistic* aspects such as the concepts of *interpretation*, *translation,* comparison of *paraphrases* and*conscious respect of rules*. In order to make pupils get aware of the role of such aspects, we introduced a fictional character called Brioshi.

3.5 Brioshi and the Algebraic Code

Brioshi is a *virtual* Japanese pupil, aged variably depending on his interlocutors' age. He does not speak any other language except Japanese, but he knows how to use the mathematical language. Brioshi loves to find non-Japanese peers for an exchange of mathematical problems via e-mail. He was introduced in order to help pupils grasp the problem of the algebraic representation of relations or procedures expressed verbally and, above all, in order to convey the idea (difficult for pupils aged 8–14) that on using a language it is necessary to *respect its rules*, which is an even stronger need when the language is formalised, owing to the synthetic nature of the symbols used. Brioshi was introduced along with structured activities: an exchange of messages to be translated into mathematical language or natural language; where the 'expert' Japanese friend plays the role of controller of the translation. If Brioshi cannot understand the translation, this must be revised through collective discussion. Such 'role-play' works with all pupils, regardless of their age, and Brioshi's arbitral role as a call to correctness and transparency brings about very positive results (for further details see Malara, 2001)

So far we have reflected on mathematical and linguistic questions about early algebra, now we shall analyse the elements which are at the basis of its methodological approach.

4 Relevant Methodological Aspects in Our Approach to Early Algebra

The didactical situations we propose are born within stimulating teaching and learning environments, but they are not easily manageable by teachers. As a consequence, those who wish to undertake innovative educational practices need to deal with a set of relevant methodological and organisational aspects that actively support a *culture of change*. We shall now discuss some of these aspects.

4.1 The Didactical Contract

The didactical contract is a theoretical construct (Brousseau, 1988) which indicates the set of relationships, mainly implicit, that govern the pupils-teacher relationship when they face the development of knowledge concerning a particular mathematical content. These relationships make up a system of obligations, involving both the teacher and his/her pupils within the teaching and learning process, which should be fulfilled and for which each of them is responsible.

In the case of early algebra, with pupils aged 6–14, the contract concentrates on the construction of mathematical conceptions rather than on technical competencies. Pupils must be brought to an awareness of the essence of the contract: they are protagonists in the collective construction of algebraic babbling. This means they should be educated to gradually become sensitive towards complex forms of a new language, through a reflection on differences between and equivalences of meanings of mathematical written expressions, a gradual discovery of the use of letters instead of numbers, an understanding of the different meanings of the "equal sign", the infinite representations of a number, the meaningful identification of arithmetic properties and so forth. In this case, the didactical contract concerns the solution of algebraic problems and is characterised by the fundamental principle '*first* represent and *then* solve'. This seems to be a promising perspective when we need to face one of the most important key points of the conceptual field of algebra: the transposition in terms of representations, from natural language (in which problems are formulated or described) to formal-algebraic language (in which the relationships they contain are translated).

4.2 The Interpretation of Protocols

Protocols are written productions made by individuals or groups with reference to a task given by the teacher. In the case of activities aimed at the enactment of algebraic babbling, constructing competencies for the interpretation of protocols and a classification of translations made by pupils implies that the teacher has to face a variety of mathematical writings, often elaborated through a mixed and personal use of languages and symbols, linked to one another in more or less appropriate ways.

Such a variety develops when the teacher stimulates, through reflection, creativity. When the pupils realise that they are producing mathematical thinking and contributing to a collective construction of knowledge and languages, they make a variety of mostly interesting and non-trivial proposals, which altogether represent a common legacy for the whole class. The collective analysis of the protocols is sharply intertwined with the practices of discussion in the classroom.

4.3 Discussion on Mathematical Themes

By 'mathematical discussion' we mean the net of interventions occurring in a class with reference to a certain situation on which pupils are requested by the teacher to express their thinking and argue in relation to what other classmates expressed as well. Through this net of interventions, the situation is analysed and debated from different points of view, until shared solutions are obtained.

The enactment of a collective discussion on mathematical themes stresses metacognitive and metalinguistic aspects: pupils are guided through a reflection on language, knowledge and processes (like solving a problem, analysing a procedure), to relate to classmates' hypotheses and proposals, to compare and classify translations, evaluate their own beliefs, make motivated choices. In this context, the teacher should be aware of the risks and peculiarities of this teaching and learning mode.

The teacher plays a delicate role in orchestrating discussions. First, he/she must be clear about the constructive path along which pupils should be guided, and about the cognitive or psychological difficulties they might encounter. From a methodological point of view, he/she must try to harmonise the various voices in the class, inviting usually silent pupils to intervene, avoiding that leaders and their followers prevail and that rivalries between groups arise. Finally, he/she must help the class recognise what has been achieved as a result of a collective work involving everybody. He/she must learn to act as a participant-observer, that is to keep his/her own decisions under control during the discussion, trying to be neutral and proposing hypotheses, reasoning paths and deductions produced by either individuals or small groups. He/she must learn to predict pupils' reactions to the proposed situations and capture significant unpredicted interventions to open up new perspectives in the development of the ongoing construction.

This is a hard-to-achieve collection of skills and a careful analysis of class processes is needed if a teacher wants to engage productively with pupils. We will shortly come back to this point, after lingering on a theoretical instrument, the glossary, which is progressively developed along with the various themes faced with teachers. Our aim is to use this instrument in order make teachers deepen their reflection on important aspects of their action in class and to become aware that, in order to achieve a good practice, they should acknowledge the value of theoretical knowledge (not only in the aspects attaining the discipline or its epistemology, but also in linguistic, psycho-pedagogical and social aspects).

4.4 The Glossary

Early algebra is a *polycentric* set of themes. When dealing with it, there is no predefined path leading to a *goal*. This can puzzle teachers, also because it is rather difficult to insert these activities in their everyday work. We believe it is necessary to outline a reference system that might help the teacher gradually achieve, through a revision of his/her knowledge, a global view of early algebra that merges *theory* and *practice*, in which *connections existing between mathematics and linguistics* are considered, in order to get closer to a conception of *mathematics as a language*.

The hypothesis is that the exploration of the glossary which is undertaken by the student teachers is an individual adventure that *depends on the way in which the teacher decides to interact with it*. Our intention is to make teachers find out a *reading key* that can promote a reflection on the *grammar* needed to explore it. Through the application of this grammar, everyone can get to know *everything they will be capable of* in that particular moment, as well as move within the polarity *local/global* along two directions: (1) within the single *local*; (2) in the *map of possible connections between the various locals*.

If we see early algebra as a *machine the functioning of which must be understood* through the re-construction of the relationships among its mechanisms, the *glossary* is its core, the constitution of which is based on the assumption that: (a) one's own knowledge is constructed by organizing exploration *in a personal way*; (b) the process of knowledge acquisition is a *constructive act* itself.

The glossary can be seen as constituted by five categories which reflect our approach to early algebra. Here are some examples (Table 1).

Each term is described by a text containing other terms in the glossary, to which it cross-refers for a wider and deeper analysis. For example, the term *Arguing* leads to *Collective*, *Process/product*, *Representation*, *Relation*, *Semantics/syntax*. We conventionally name this set of terms *the Net* of the term *Arguing*. Generalising, a Net of a term is 'the set of cross-references that link the term with other terms of the glossary'. So the Glossary can be seen as a matrix of Nets.

Table 1 Five categories of an approach to early algebra

Categories	General	Pre-algebraic thinking, relational thinking, process/product, metacognitive/metalinguistic, Brioshi, representation, represent/solve, opaque/transparent representations, mediation tools ...
	Mathematical	Formal coding, additive/multiplicative form, mathematical sentence, unknown, regularity, relation, structure, equal, variable, ...
	Linguistic	Arguing, algebraic babbling, canonical/non canonical form, letter, language, metaphor, paraphrase, semantics/syntax, translate, ...
	Social-educational	Collective (exchange, ...), sharing, didactical contract, discussion, social mediation, negotiation, ...
	Psychological	Perception, emotional interference, semantic persistence, ...

General, psychological and social questions represent a *methodological support* to mathematical and linguistic questions. Although early algebra concerns *mathematics* education, it is certainly true that the importance of the three ‘supportive’ components is fundamental. The approach to arithmetic in an algebraic perspective is based on a strong basis of *social* and *psychological* assumptions, and on a set of *general* basic concepts that the teacher should learn to promote and manage.

In order to carry out mathematical activities, teachers cannot forget that (a) the construction of knowledge occurs through promotion of *social dynamics* that favour *exchange* and *verbalisation* in the classroom; (b) the identification of suitable *didactical mediators* (such as Brioshi) is crucial to a stable acquisition of *meanings*; (c) it is necessary to promote activities highlighting *metacognitive* and *metalinguistic* aspects.

All these interrelationships are highlighted in the glossary by the nets of its terms and by numerousness of the occurrence of each term in the nets.

5 The Present Study and Its Methodology

Our research experience with the teachers made us aware of their difficulties as to the designing and managing classroom discussions (Malara 2003, 2005). This persuaded us about the importance of the analysis of classroom processes in order to help teachers acquire the necessary competencies for the orchestration of discussions. For this reason, in these past few years we addressed our research towards the individuation of methodologies and tools aimed at producing in teachers the mathematical/pedagogical competencies necessary to approach early algebra in a socio-constructivistic way.

Our research methodology is based on: (1) planning didactical classroom routes with the teachers; (2) teachers’ production of *diaries*, i.e., transcripts of audio-recordings of the classroom activities intertwined with their comments and reflections); (3) joint (researcher & teacher) analysis of diaries; (4) sharing of the diaries among the teachers involved, writing of meta-comments, discussions and reflections.

Documenting the classroom episodes analysis – paradigmatic in highlighting the sharp correlation between the students’ mathematical constructions and the teacher’s sensitiveness in the discussion – brought us to elaborate the model of task we focus on in the following paragraphs.

6 A Model of Task for Teachers: The Analysis of Classroom Scenes in Sequence

Our model of task gives teachers the chance to deal with the practice of constructive teaching ‘theoretically’, forcing them to focus on provisional and reflection-related aspects. The task develops along *5 or 6 Scenes*, structured as a ‘connected set of issues', and a *Final Reflection*. The scenes are based on excerpts of transcripts of one of our experimentations. Each scene is composed of two sections: the first concerns the presentation of a classroom situation, and the second focuses on questions for the teacher

Teachers are sequentially proposed scenes at regular time intervals (of about 20–30 min): while they are working on the first scene they do not know the second yet; when they work on the second one they do not know the third one yet and so forth up to the conclusion. After analysing the input proposed in the first part of the scene, the teacher makes hypotheses about the class' reaction. In the subsequent scene these hypotheses are compared with what actually happened and so forth up to the last scene. At the end, a global review of the work carried out is formulated.

This kind of task has developed with time and can be seen as the result of a research process. The first idea was born of producing interconnected e-learning worksheets for teachers in order to promote a constructive and linguistic approach to algebra, to which a transposition of these materials in a first version of the task for teacher workshops followed. The present version of the task has been used in workshops for teacher-trainees of junior secondary school since 2005. The example we report was given to 45 teachers as the final test of a 40-h course on didactics of early algebra. Here, 20 h were devoted to workshops dealing with similar tasks together with the analysis of excerpts from classroom discussions.

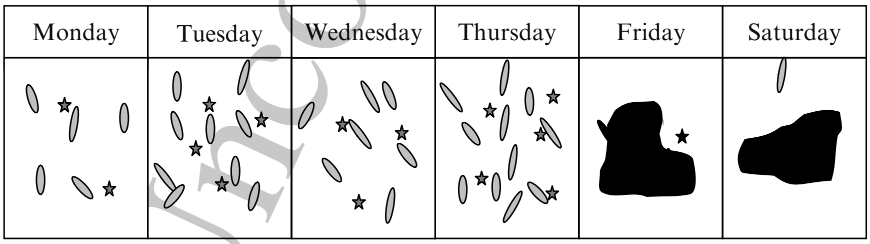
6.1 An Example of Task in Early Algebra

The First Scene

The class teacher presents the situation:

Ann likes chocolate cookies and finger biscuits, which she eats for breakfast every school day. She eats different quantities every day, but follows a rule she set.

Then she shows a drawing with two ice cream spots which are hiding only the finger biscuits Ann ate on Friday and the chocolate cookies she ate on Saturday.



At this moment the teacher gives a first task: Write down Ann's rule in natural language.

Task for teachers:

1. Carry out the task.
2. Give a short explanation of why it is important to search for regularities in mathematics.
3. Discuss the instruments and/or strategies you view as the most effective to identify regularities.

One initial goal is to make teachers identify possible difficulties pupils might encounter and think about instruments and strategies that can be used to solve the task. By examining the drawing, pupils must find a link between the number of biscuits and that of cookies. This is not a difficult task if one tries to express the number of biscuits as a function of that of cookies; much more difficult is to express the relationship in the inverse way. Other two aspects, that should not be underestimated, concern the issues of the interval in which each quantity may vary and the generalisation of the law (the situation might induce a cyclic view of the law and inhibit its view in general terms).

Saturday is problematic because, once identified the law in the formulation ‘the number of finger biscuits is one more than twice the number of chocolate cookies’, the solution is that Ann does not eat any chocolate cookie on that day. From a psychological point of view, this brings about a conflict with the implicit hypothesis that Ann always eats both types of biscuits, since one can infer that she loves both. From a mathematical point of view, this entails the acceptance of zero as a number (since zero is an indicator of absence of quantities).

More in general, we aim at verifying the impact of theoretical studies developed during the course and the use of specific constructs, either of mathematical character, such as ‘canonical and non canonical form of a number’, ‘procedural-relational polarity’, or of educational-methodological character such as ‘didactical mediator’ or ‘didactical contract’.

The Second Scene

Pupils’ sentences are classified by the class teacher and the most representative ones are written on the blackboard. Then the teacher opens up the discussion with the purpose of making the class decide what sentence best represents the rule followed by Ann.

(a) Ann takes from finger biscuits the same number of chocolate cookies, she multiplies it by 2 and adds 1.
(b) She eats an odd number of finger biscuits and from 1 to 5 chocolate cookies.
(c) Chocolate cookies are always one more.
(d) One day she eats more and one day she eats less

Task for teachers:

4. Comment upon each of the four sentences written on the board concerning their consistency, completeness and efficacy with respect to the formulation of the regularity.
5. Imagine a plot for a discussion: opening, key steps and end.

This scene makes teachers face the task of first analysing prototypes of pupils' productions, trying to trace back the mental views that produced them and interpret unexpressed intuitions; and second imagining the development of a possible class discussion aimed at sharing results and constructing a clear proposition expressing the law, so that the subsequent task of translating it into algebraic language becomes easier.

The Third Scene

After a collective discussion, pupils agree on the choice of (a).
The class teacher proposes '*Each of you may try to write (a) in other ways'*
After a while some significant expressions written by pupils are reported on the blackboard:

(a_1) The number of finger biscuits is 1 more than twice the number of chocolate cookies.
(a_2) The number of finger biscuits is twice plus 1 the number of chocolate cookies.

Based on these sentences the teacher opens up a new discussion.

Task for teachers:

6. Comment the choice made by the class to propose sentence (a).
7. Analyse the last three sentences, highlighting any possible difference with reference to the relational-procedural polarity.
8. Figure out a plot for a possible classroom discussion.

The aim is to lead teachers to refine their interpretative skills in the case of verbal expressions produced by pupils in order to guide them to identify differences, by making them clear: (a) in which sentences the time dimension of the counting act (procedural view) prevails over those in which the relationship between the two quantities is objectified; (b) how the interaction of these two views induces a verbal formulation of the law in which the predicate 'to be', typical of relational formulations, is re-formulated in terms of equality ('to be equal to').

The Fourth Scene

At the end of the discussion the class chooses sentence (a_1); the remaining two are erased.

The teacher gives a second task: Translate Ann's law for Brioshi in algebraic language.

Task for teachers:

9. Translate Ann's law in more than one way.
10. Predict the possible translations by pupils.

The goal of this task is twofold: first it is to evaluate teachers' abilities in finding algebraic formulations equivalent to the literal translation of the sentence; second, and this is more delicate, to predict pupils' translations, including the naive or incorrect ones.

The Fifth Scene

After an individual work, sentences are written on the blackboard (pupils' legend in brackets):

1) 1×2	2) $a + 1 \times 2$(a = number of finger biscuits)
3) $fb + 1 \times 2$	4) $a \times 2 + 1$
5) $fb + 1 \times 2 = a$	6) $b = c + 1 \times 2$
7) $a \times +1 = b$(a = number of chocolate cookies)	
8) $a = b \times 2 + 1$	9) $(a - 1) \times 2$

A discussion is enacted to choose the sentence that should be sent to Brioshi.

Task for teachers:

11. Comment upon each translation of Ann's law, underlining their correctness/incorrectness, consistency, possible redundancies etc.
12. Predict what the class will possibly choose and the related argumentation about the sentence they will send to Brioshi.

A delicate issue in this scene is the comparison between the expressions: '$a = b \times 2 + 1$' and '$a \times 2 + 1 = b$ (a = number of chocolate cookies)': the same letters represent different variables and it would be appropriate to predict pupils' behaviour on facing this fact.

The Sixth Scene

At the end of the discussion about the sentence to be sent to Brioshi, the following rule is chosen: $b = a \times 2 + 1$. The class teacher poses the problem *Are we able to understand what happens on Friday and Saturday even if ice cream* *spots hide part of the drawings?* Some pupils almost immediately use the formula correctly, but other pupils are initially puzzled. This difficulty is overcome during the discussion.

Task for teachers:

13. Is the rule sent out by the class the same you predicted at the end of the Fifth Scene? Is it different? Write down your comments on this.
14. Identify what the spots hide in the table illustrating the biscuits Ann eats.
15. With relation to the question posed by the teacher, interpret the reasons underlying the widespread confusion in the class.

This scene is a chance to test teachers' abilities on both the didactical and the mathematical side: teachers are expected to argue about mental and operative processes that pupils need to enact and about difficulties brought about by the Saturday situation, linked to the interpretation of zero as the solution to the equation '$a \times 2 + 1 = 1$'.

Concluding Reflections

16. Write down a short reflection on the didactical situation you were asked to comment upon, also referring to a possibility of reproducing it in a hypothetical class of yours.
17. Write down a short concluding reflection on the structure of the whole set of tasks, mainly referring to its significance as model of task that may help trainee teachers explore what they learned both at mathematical and pedagogical level.

This is a task where teachers are expected to express themselves about the reproducibility of the didactical situation and, more generally, on the global value of the task they just carried out, with reference to their culture as to 'pedagogical content knowledge' (Shulman, 1986).

7 Analysis of Teachers' Protocols

7.1 Study of the First Scene

Question 1. *Carry out the task*

Eighty percent of the teachers easily identify the correspondence law. Two different kinds of behaviour arise: some identify the law by analysing numerical cases following the days' order; others feel the need to rank data according to the increasing number of chocolate cookies. In both categories there are some who make comments that highlight their view on the situation:

> Day by day Ann eats a number of finger biscuits which is twice the number of chocolate cookies plus one finger biscuit more. There does not seem to be a 'defined' number, expressing a certain 'rule' for the choice of chocolate cookies[3] (I can only see that chocolate cookies increase from 0 to 11). The relationship between finger biscuits and chocolate cookies only links the number of finger biscuits to twice plus 1 the number of chocolate cookies. Moreover, I observe that the numbers of finger biscuits are all odd.

The underlined sentence reveals the fact that the author sees the relationship as limited to the examined cases and not as a general one. In the latter case, in fact, the set of natural numbers would be seen as the domain in which the number of chocolate cookies varies (this is highlighted by other protocols). Though an acknowledgment of a realistic limit to the number of cookies that can be eaten might be reasonable.

Finally, about 20% do not interrelate the two variables but consider them separately:

> On each day Ann eats 1 chocolate cookie more than two days earlier and two finger biscuits more than two days earlier.

The difficulties met by these teachers show their little familiarity with the identification of correspondence laws, and, at the same time, how complex it is for inexperienced subjects to work in the 'search for regularities' environment.

Question 2. *Give a short explanation of why it is important to search for regularities in mathematics*

Teachers did not always express their ideas relevantly:

> Searching for regularities is important because it makes order arise from an apparently chaotic set, thus giving opportunities to find the relationship between entities of the set itself.

Many underline the importance of searching for regularities in order to favour the transition from the particular to the general, or sometimes of determining laws that can be adapted to other contexts or situations. A minority underlines the predictive power of laws with relation to the studied phenomenon. Some others highlight the educational value: through this activity pupils can be led to understand how mathematical laws originate:

> Searching for regularities is important because it enables us: to make the process leading to the formulation of any mathematical law more transparent; to help pupils to develop the metacognitive thinking which enables them to abstract and generalise.

Question 3. *Discuss the instruments and/or strategies you view as the most effective to identify regularities*

Many teachers refer to the rewriting of the table according to increasing values of the number of chocolate cookies as a helpful strategy for the identification of the

[3] Our underlining.

law and its generalisation. Few teachers mention the class discussion as a strategy that may favour a sharing of the law's verbal formulation (a step we consider important towards its algebraic coding). Often, notions learned during the course emerge, for instance, the reference to non-canonical representation as a tool to make the correspondence between pairs transparent and to get to the generalisation of the law. An example:

The situation is suitable to a collective discussion in class. After listening to pupils' descriptions in natural language, I would propose them to construct a table at the blackboard in order to report numerical values in an organised way:

No. of chocolate cookies	2	4	3	5
No. of finger biscuits	5	9	7	11

At this point it would be important to invite pupils (even after they understood the relationship between the number of finger biscuits and that of chocolate cookies) to "paraphrase" the canonical form of the number of finger biscuits into the non-canonical form , i.e. to see it in a less opaque form. In fact, the number of finger biscuits might be seen in this way, n. finger biscuits:

$$5 = \underline{2}'2 + \underline{1};\ 9 = \underline{2}'4 + \underline{1};\ 7 = \underline{2}'3 + \underline{1}$$

So pupils might be able to identify "blocked" numbers (2 and 1). The last step would be getting to the generalisation: $m = 2'n + 1$ *with* m *= number of biscuits and* n *= number of cookies*.[4]

8 Study of the Second and Third Scene

The two scenes highlight the educational importance of the verbal formulation of the observed regularities. Teachers often underestimate it, not grasping its impact on the algebraic formulation of a law. This often happens because teachers do not have a linguistic view of the approach to algebra.

Question 4. *Comment upon each of the four sentences written on the board as to their consistency, completeness and efficacy with reference to the formulation of the regularity.*

Question 6. *Comment upon the choice made by the class to propose sentence (a).*

Question 7. *Analyse the last three sentences, highlighting any possible difference with reference to the relational-procedural polarity.*

Some protocols mainly highlight the comparison of 'relational representations' and 'procedural representations': sentences a_1 and a_2 are identified as relationships, whereas a_3 is seen as strictly procedural. Some others reveal sensitivity towards important but barely noticed aspects, possible outcome of what has been done during the course:

[4] The underlined words reflect the study of theory: each of them is a specific item of the Glossary.

> I would say that from a_1 to a_3 we can notice an increasing 'unclearness of thinking' and a use of natural language that hides a more and more elementary mathematical thinking. A_3 denotes a clear 'arithmetic[5]' approach by the pupil; it may be translated into a classical operation read from left to right with the result on the right, after the equal sign.

In other ones, we find explicitly stated difficulties met by the teacher or his/her beliefs translated as 'pupils' inclinations or difficulties':

> I think that I would hardly convince pupils that sentence a_1 is better than a_3 because a_3 is the most simple to be translated and I think it is more transparent and clear to them because it carried inside the exact procedure they should apply.

Many analyse sentences referring to the difficulties inherent in their algebraic formulation:

> The source sentence 'a', although correct, is not very clear but was helpful for pupils in order to elaborate more transparent sentences: a_1) 'The number of finger biscuits is 1 more than twice the number of chocolate cookies' can be translated by following the sequential nature of the sentence into: m = 1 + 2n; a_2) 'The number of finger biscuits is twice plus 1 the number of chocolate cookies' can be translated into: m = 2n + 1.; a_3) 'The number of chocolate cookies multiplied by 2, adding 1 equals the number of finger biscuits' is translated into: 2n + 1 = m.

Question 5 *Imagine a plot for a discussion: opening, key steps and end*

Question 8 *Figure out a plot for a discussion*

In these situations teachers have great difficulties; besides those who ignore the request or simulate scarcely constructive excerpts of discussions, teachers' behaviour can be grouped into four categories. Teachers:
(*a*) simplify the task restricting it to a generic talk about what needs to be done:[6]

> The discussion starts from the analysis of the four proposed sentences; the goal will be to identify the most transparent one, the one which better describes the situation by means of arguments produced by each pupil (each will possibly provide arguments supporting their own sentence). The outcome should be the choice of the first sentence (a), as the one which best reflects Ann's law.

(*b*) only figure out the opening of the discussion:

> the first situation might be analysed asking pupils to represent it and then making them reflect on the meaning of 'multiplying by 2' and 'adding'

(*c*) sketch out the discussion's structure, focusing on themselves as teacher:

> I would start from (c) and I would remark that what the pupil says is not true. To get to an organised analysis I would suggest the construction of a table (key step). I would say the same in the case of (b) and say that to be sure that finger biscuits are always odd we must find a rule that enables us to calculate them every time, then I would point out that sentence (a) generalises what happens every day and it is the only one which enables us to see what happens on Friday and Saturday.

[5] Several students use the term 'arithmetic' with the meaning of procedural.

[6] All protocols in this paragraph refer to the second scene, except where differently specified.

(d) sketch out the discussion's structure, focusing on students:

In a hypothetical discussion pupils might be led to reflect starting from sentence (c); they would notice that not chocolate cookies but finger biscuits are one more. Once this point is made clear, the reflection might shift on (b), pointing out that it is not sure that on days like Friday, Saturday and Sunday Ann eats a number of c. cookies between 1 and 5, keeping the fact that the number of f. biscuits is odd. Next, sentence (d) might be analysed, highlighting that it is not a rule that clarifies the relationship between the number of sweets (c. cookies and f. biscuits). Finally I would focus on (a) and analyse it from the point of view of meaning, also using additional strategies (e.g. graphical, like a table) making pupils get, through an argument-supported progressive reflection, to say that (a) is the one that best expresses Ann's law.

9 Study of the Fourth Scene

In this scene the central problem is the formal translation of the relationship 'the number of finger biscuits is 1 more than twice the number of chocolate cookies'. The task for teachers is twofold:

Question 9. *Translate Ann's law in more than one way*

Question 10. *Predict the possible translations by pupils*

Behavioural categories emerged can be classified as follows:

(a) Poor productions: teachers do not distinguish the different levels in the two tasks and simply produce the translations '$2'\mathrm{c}+1=\mathrm{f}$' and '$\mathrm{f}=1+2'\mathrm{c}$' ('f' representing the number of finger biscuits and 'c' that of chocolate cookies) and declare for example:

> The first one can be the most frequent among pupils because the equal sign is at the end.

(b) Rich productions, centred on teachers only: teachers translate the relationship also in the implicit form, or taking the number of chocolate cookies as subject (a translation that pupils can hardly make in this phase); they do not put themselves in pupils' shoes and are not able to predict possible translations;

(c) Rich productions, centred on teachers and pupils: teachers provide a wide range of translations and at the same time analyse them in terms of possible pupils' translations. As to this, teachers' hypotheses differ considerably. They maintain that:

(c_1) the most probable translation is the one that represents the calculation process to get the number of finger biscuits ($2'\mathrm{c}+1=\mathrm{b}$);

(c_2) the most probable translation is the literal one ($\mathrm{b}=2'\mathrm{c}+1$);

(c_3) both can be chosen, as the following protocol claims:
u = n. chocolate cookies; v = n. finger biscuits: (1) $u'2+1=v$; *(2)* $v=1+u'2\ldots$

> I think that among the possible translations the first one would be the most popular because it puts equal at the end, but also the second one could be very spread because it follows the literal and sequential translation of the sentence in natural language.

As to Question 10 in particular, ‘*Predict pupils’ possible translations’,* almost all teachers suppose that pupils:

- in translating the term ‘twice’, would make explicit use of the multiplication sign ‘′’ and that weaker pupils would use the additive representation;
- would use writings referring to the semantics of the situation, such as ‘$\mathrm{nb} = 1 + 2'\mathrm{nc}$’
- omit the explanatory key to the meaning attached to each letter used.

One controversial point concerns the possible making of ‘twin’ representations such as ‘$\mathrm{b} = 1 + 2'\mathrm{c}$’ and ‘$\mathrm{b} = 2'\mathrm{c} + 1$’; the commutative law is intuitive to some teachers but not to others. Few teachers reflect on the possible wrong or incomplete translations.

10 Study of the Fifth Scene

Question 11. *Comment upon each translation of Ann’s law, underlying their correctness/ incorrectness, consistency, possible redundancies etc.*

Almost all teachers carried out a careful analysis of the sentences, justifying also the possible reasons that led to them. What comes out is a ‘cultured’ reading, making reference to theory. Terms like ‘adulterate’ or ‘unfaithful’ translation which were introduced at a theoretical level during the course, drawing on linguistics[7], were used.

Question 12. *Predict what the class will possibly choose and the related argumentation about the sentence they will send to Brioshi.*

As usual, this is the most problematic task. Not all teachers carried it out successfully; on the basis of the analysis, many of them only indicate the following sentences as the one that will be chosen by the class: ‘$\mathrm{a} = \mathrm{b}'2 + 1$’ and ‘$\mathrm{a}'2 + 1 = \mathrm{b}$, a = number of chocolate cookies’. In the first translation, more faithful to the text, the meanings of the used letters are not expressed; in the second one the problem is to lead pupils to reflect upon the symmetrical use of the equal sign. There are also teachers who express these concepts by making up brilliant excerpts of possible conversations in the classroom.

The effects of training can be seen in those who tackle the problem of the collective analysis of the various translations: the courses they followed highlighted the importance of making pupils interpret ‘unclear’ sentences in order to get to a reasoned choice of the correct ones, appreciating everybody’s contributions. About this, a teacher writes:

> … We might also point out that all sentences are not wrong ‘per se’, as they indicate something anyway, but they do not reflect the presented situation.

[7] For a deeper discussion about these aspects see Malara and Navarra (2001).

Somebody conjectures that initially some pupils may make 'easy' choices because they are semantically more expressive:

> Initially some might be 'attracted' by sentences that make use of abbreviations because they are more 'pleasant' from the symbolic point of view, and exclude them only later.

Another important emerging aspect concerns teacher's conception of their own role in the development of a class discussion:

(a) some view the management of the activity as giving room to pupils' interventions:

> The class might initially take into consideration sentences between the fifth and the eighth, because only in them is the number of finger biscuits compared to the number of chocolate cookies. By substituting numbers in each translation[8] the class soon realises that only the seventh and the eighth translations are valid. Perhaps in the end the class would choose to send the seventh sentence because it best translates the starting verbal expression.

(b) some put themselves at the centre and view the class in a listening attitude:

> I think that after listening to the various translations, the class needs to decide between '$a = b'2+1$' and '$a'2+1 = b$. (a = number of chocolate cookies)'. Both are written in correct mathematical language and represent the situation consistently. Nevertheless, I believe that $a = b'2+1$ is more precise, because it respects the sequence and order in which the law is expressed.

11 Study of the Sixth Scene

In this scene, teachers face the following questions:

Question 13. *Is the rule sent out by the class the same you predicted at the end of the Fifth Scene? Is it different? Write down your comments on this.*

Question 14. *Identify what the spots hide in the table illustrating the biscuits Ann eats.*

Question 15. *With reference to the question posed by the teacher, interpret the reasons underlying the widespread confusion in the class*.

Puzzlement refers to the study of Friday and Saturday cases, in which it is necessary to substitute the known value of the variable in the given law and solve the resulting equation. In order to do this, teachers must re-examine the initial situation and highlight that the Saturday case concerns the inverse relationship and that an equation is needed to solve it. Teachers mainly give correct interpretations:

[8] The teacher thinks of the number substitution as a suitable strategy to verify the correctness of formal translations.

> One common difficulty may be linked to Saturday, because it is difficult to accept that Ann does not eat any chocolate cookie on that day. Another possible reason may attain to Saturday from an operational viewpoint: pupils have the number corresponding to 'b' available and must find 'a', and this is a non trivial task for grade-6 pupils.

Some interpretations are particularly meaningful, as they are conducted in the line of a co-ordination of cognitive and psychological aspects. For example:

> I think the difficulty was ... also to accept the idea that in the graphical representation the ice cream spot did not cover anything!

12 Study of the 'Concluding Reflection'

Question 16. *Write down a short reflection on the didactical situation you were asked to comment upon, also referring to a possibility of reproducing it in a hypothetical class of yours.*

Question 17. *Write down a short concluding reflection on the structure of the whole set of tasks, mainly referring to its significance as model of task that may help trainee teachers explore what they learned both at mathematical and pedagogical level.*

The productive presentation of the activity as a playful challenge is underlined and the general educational value of the game is analysed. Teachers also point out how attractive the game can be, due to the arguments solicited and to the mediation towards algebraic language by means of natural language:

> This is certainly a didactical situation that will be fun for the class, also because it leaves pupils free to express themselves and the "heaviness" of reasoning in algebraic terms does not come up immediately. Playing games is always helpful.

Generally positive opinions are expressed about the reproducibility of the proposed situation, even though some envisage an unsuccessful management of the class:

> I would certainly propose this kind of situation (I believe this teaching approach is very good and for sure to be preferred to traditional teaching); my fear, perhaps due to lack of experience, is to be unable to deal with unexpected situations which come out in the class' dynamic processes.

Reflections gather around two aspects, often interwoven: (a) value of the task as guiding instrument for the teacher; (b) complexity of the task, difficulties and fears this generated. We deal with the two aspects separately.

(a) ***Value of the task as guiding instrument for the teacher***

It is generally acknowledged that the task is a valid instrument to lead teachers, and especially those who do not have class teaching experience, to approach issues that characterise the teaching activity such as predicting pupils' behaviour, interpreting their productions and setting up discussions.

> The type of task is very positive. Not knowing what happened next, I was forced to think more and examine all the possibilities I could think about. The subsequent scenes can be used to confirm or refute reasoning and permit a meta-analysis of what we developed and appropriated as ours in the previous scene.

The fact that the task forces the teacher to predict pupils' behaviour is highlighted:

> This test can be helpful to put ourselves in pupils' shoes, to consider aspects that are usually taken for granted and predict 'unusual' developments (often pupils react in ways we did not expect).

Several teachers underline that the task is presented in a very stimulating and attractive form:

> I never entered a school as a teacher and my memory as student focused on a traditional school made of rules and definitions given from above! Anyway, I believe that dialogue and discussion are the foundations of learning and engaging with this didactical situation was very instructive and amusing.

Others recommend that this kind of tasks be proposed more often:

> I think that in the professional training of teachers, situations of this kind should be proposed very often during the course in the form of laboratories.

(b) ***Difficulties met and fears raised***

Some teachers underline the complexity of the test, highlight difficulties and most of all express their fears about their own abilities to carry out constructive teaching in the future. The difficulties teachers express are grouped as follows:

(1) Difficulties met in carrying out the task (trainee's point of view about the task)

> As you might have observed I found it difficult to find the rule. ... I must say that I would probably not have obtained good results after proposing classes in this way, because, as we said during the lessons, it is up to pupils to propose the various ways of translating a sentence and moreover they all must be ready to acquire this type of language.

(2) Wrong perception of the situation's impact on pupils

Teachers coming from background studies in physics or mathematics underestimate the difficulties that students may encounter:

> The search for regularities may be used because it enables an easy transition from a concrete example to its abstraction and because it allows for simple translations in mathematical language, thus allowing the subject to grasp relationships between elements.

(3) Difficulties in predicting one's own actions towards students (trainee's point of view as to students)

> The main difficulty comes up when pupils propose their situations in symbolic terms. If their answers are partially predictable, it is sometimes difficult to get into their reasoning and understand which thought led them to elaborate certain pseudo-equations.

(4) Difficulties due to the intersection of the role of teacher and that of trainee

Trainees often are temporary teachers, this double role becomes an object of reflection:

> I start with this self-critical reflection. I admit I misread the expression reported on sheet 1 '... two ice cream spots hide finger biscuits on Friday and c. cookies on Saturday' and therefore I ignored its meaning (I believed that biscuits outside the spot were not relevant). ... I suppose at this point I transmitted the original text of the problem in a wrong way to the class, making pupils puzzled and misleading their understanding. In the case none of them would have asked for explanations we would have naturally reached the sixth scene with puzzlement and incapacity of completing the tasks, as it actually happened.... Nevertheless I fell into this trap because today I suffered the 'test' almost like a pupil and that probably would not have happened if I had studied and proposed it as a teacher.

(5) The key point for the management of a class discussion

Protocols clearly show that letting pupils discuss is a source of worries for teachers, especially due to their lack of expertise:

> ... What worries me more (and I scarcely analysed it in this test) is how a collective discussion should be managed and how it is possible to lead the class to share correct meanings. I feel that the goal of each proposed phase is rather clear to me; what is difficult is to actually lead the class to that, by guiding the discussion in a constructive way, without harsh criticisms or forcing.

One problem concerns the impact of the primitive model of teacher in the classroom as far as the management of the discussion is concerned. Sometimes teachers, although agreeing to tackle a discussion, figure out a directive way of piloting it:

> I would set up the discussion starting from the last sentence back to the first one, trying to make pupils understand that the last two ones are useless to our aims, that the second one only expresses a partial truth and that the first one is the only one that expresses a useful regularity.

13 Concluding Remarks

In this chapter we sketched a synthesis of our theoretical framework aimed at an early and linguistic approach to algebra, to be carried out constructively. We recalled some of our theoretical constructs, including 'algebraic babbling', which reflects our conception of algebraic language learning. We underlined how valuable for teacher education are the study of theory and the reference to a glossary, instrument which comes to fulfil the constantly increasing intellectual need (Harel, 1998) felt by teachers.

We tackled the issue of how teachers can be led to acquire conceptions and behavioural models that may foster among pupils a view of algebra as a representation system, and in particular develop in pupils skills related to modelling, interpretation and production of thinking.

For this reason we proposed here our model of task, specifically designed so as to give teachers the chance to deal with the practice of constructive teaching ‘theoretically’.

The structure of the task, divided into scenes, is considered an effective classroom simulation to guide the teacher in a step-by-step involvement with the teaching sequence. Hypotheses about classroom-based actions are seen as very demanding tasks, still it is highlighted that the task is a powerful means to force the teacher to compare predictions with the actual development of the class activity.

Protocols show teachers’ difficulties and fears concerning their own future abilities to implement constructive teaching. Difficulties mainly concern the design of discussion plots, the interpretation of and comment on formal written expressions, identification with students.Other fears concern not being able to understand pupils, confusing them, not being able to make them actually discuss, and implementing a directive way of piloting the class due to an unconscious stereotyped model of teacher.

The protocols also highlight remarks about teachers’ awareness of how culturally important it is to make their own development as teachers explicit and open to collective sharing. Processes in which new knowledge and new meanings are shared, come to be enacted at three different levels:

- with oneself, leading to reflections on the courses and the tasks impact on ones own professional development and, more generally, on the ‘renewed’ relationship between theory and practice;
- with colleagues, by means of a peer to peer comparison of things learned during the course and the explicit statement of underlying beliefs, this leads teachers to become aware of the features of their own beliefs as well as of their way of living the classroom activity (the latter is a fundamental step for a review of one’s own attitudes and a change of unsuitable ones);
- with pupils, maybe the most stimulating and innovative aspect- as it induces teachers to set up a didactical contract based on constant explicit claims about the reasons underlying the teaching activities they propose to pupils. This way, they get used to seeing themselves as active participants in the construction of knowledge.

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