Future teachers facing proof problems: Games of interpretation, anticipating thought and coordination between verbal and algebraic register[1]

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ABSTRACT

This contribution is part of a wider work, involving pre- and in-service teachers, aimed at making teachers aware of the importance of developing in students an effective symbol sense. In this perspective, a fundamental part of our work with teachers consists of activities of proof in elementary number theory. At the same time we are interested in analysing the use and the role of algebraic language in the development of such proofs. In this paper we present an initial analysis and classification of trainees' behaviours in facing the proof of a conjecture that arises from their exploration of a proposed numerical situation. The analysis of teacher protocols has been conducted by reference to particular interpretative keys that reflect our thinking about the abilities required for the deployment of algebraic language in proofs. They are: the application of specific conceptual frames, the games of interpretation between different frames, anticipatory thoughts, the use of conversions and treatments and coordination between different registers of representation. From an educational point of view this analysis can be used as a methodological tool to both help teachers identify their own difficulties in algebra and to promote communication with and amongst them.

Key words: Algebra, Proof, Symbol Sense, Conceptual Frames, Games of Interpretation, Anticipatory Thoughts, Registers of Representation, Teacher Education.

1. Why proofs in elementary number theory?

As observed by Kieran (2006), "*the question of meaning lies at the heart of research in algebra*". Many research studies support an approach to algebraic language related to the development of reasoning.

The fundamental theoretical tool of reference for our work is the concept of symbol sense developed by Arcavi (1994, 2005), who claims that, in spite of their abilities in the manipulation of algebraic expressions, students often do not see the value of algebra as an instrument for the understanding, expression and communication of generalizations, the establishment of connections, or the production of argumentation and proof.

The author chooses to avoid explicitly defining symbol sense in favour of highlighting, through meaningful examples, the attitudes to stimulate in students to promote an appropriate vision of algebra.

Particular attitudes that he names include: the ability to know when to use symbols in the process of finding a solution to a problem and, conversely, when to abandon the use of symbols and to use alternative (better) tools; the ability to see symbols as sense holders (in particular to regard equivalent symbolic expressions not as mere results, but as possible sources of new meanings); the ability to appreciate the elegance, the conciseness, the communicability and the power of symbols to represent and prove relationships).

[1] Work carried out in the European Project 'Transforming Mathematics Education through Teaching-Research Methodology'.

For these reasons, Arcavi argues that students should be introduced to algebraic symbolism from the beginning of their studies through activities that encourage in them an appreciation of the value and power of symbols for describing arithmetical phenomena and expressing generalizations, and as tools for understanding, solving and communicating problems.

Many researchers share a similar vision of the approach to the teaching. Among them, Bell (1996), for example, states, in particular, that it is necessary to favour the use of algebraic language as a tool for representing relationships, and to explore aspects of these relationships by developing those manipulative abilities that could help in the transformation of symbolic expressions into different forms. Similar observations are also found in Wheeler (1996), who asserts the importance of ensuring that students acquire the fundamental awareness that algebraic tools "open the way" to the discovery and (sometimes) creation of new objects. Kieran (2004) also stresses the importance of devoting much more time to those activities for which algebra is used as a tool but which are not exclusively to algebra (global/meta-level activities according to Kieran's distinctions). These activities help students developing transformational skills in a natural way since meaning supports manipulations (Brown 2004).

Proof is certainly one of the main activities through which helping students develop a mature conception of algebra. Wheeler (1996) states that activities of proof construction could constitute "*a counterbalance to all the automating and routinizing that tends to dominate the scene"*. Selden and Selden (2002) argues that Elementary Number theory is "ideal for introducing students to reasoning and proof" because it makes students deal with familiar objects and reduces the level of abstraction required.

As Zaksis and Campbell (2006) state that "the idea of introducing learners to a formal proof via number theoretical statements awaits implementation and the pros and cons of such implementation await detailed investigations" (p.10).

We believe that activities of proof in elementary number theory would both provide students with the opportunities they need to progress gradually from argumentation to proof and help them to appreciate the value of algebraic language as a tool for the codification and solving of situations that are difficult to manage through natural language only (Malara 2002). In fact, many different competencies are required of a student who has to face proof problems in elementary number theory. In particular, he/she has to: (a) know the meaning of the mathematical terms in the problem text and interpret them correctly by reference to it; (b) translate correctly from the verbal to the algebraic language; (c) be able to interpret the results of the transformations operated on the algebraic expressions in relation to the examined situation; and (d) control the consequences of his/her assumptions. The teacher as model plays a fundamental role in helping students to acquire these competencies. Through the selection of appropriate situations, he/she needs to show them how to translate hypotheses into algebraic language, how to transform an expression to find the range of its possible interpretations, how to interpret the results of these syntactic manipulations, and how to select the expression appropriate to the thesis.

Therefore, Mathematics teachers preparation requires also to develop abilities in using algebraic reasoning effectively for proof in Elementary Number Theory, as also stated in the book 'The Mathematical Education of teachers', edited by the CBMS (Conference Board of the Mathematical Sciences). Our work with and for teachers, which is elaborated within this schema, focus on this central aim.

1.1 Methodology of work with pre- and in-service teachers

How can a teacher be prepared to help his/her students develop an effective symbol sense if his/her own vision of the teaching of algebra is incomplete or lacking? (On future teachers' conceptions of algebra see, for example, Malara 2003). Teachers are unlikely to be good models for their students without being aware of the limitations of their own vision of algebra and of how to teach it.

Consequently, as part of our postgraduate course in algebra for secondary school pre- and in-service novice teachers we offer a cycle of workshops, subdivided into two hours sessions (20 hours in all), designed to encourage the trainees to evaluate their own knowledge and views. The workshops are structured in three phases: 1. individual trainee work; 2. researcher analysis and classification of trainee protocols; 3. collective discussion of the protocols.

Among the proposed activities, those concerning proof in elementary number theory are considered fundamental. A range of proof problems are assigned to trainees during subsequent phases of work. In the perspective of the cognitive unity of theorems (Boero *et Al.*, 1996), every problem we pose them starts with the formulation of a conjecture based on numerical explorations and then proceed to its prove. Trainees are required to solve a range of problems and then analyze them in terms of the difficulties they consider students are likely to face. Subsequently trainees are asked to analyze occurrences of student attempts at the same proofs from the solutions of the same problems and to produce an analysis of, and suggest possible reasons for, the students' mistakes and blocks.

The problem we present in this work is part of an initial test, proposed to a group of 23 secondary school pre- and in-service novice teachers (8 physics, 9 mathematics and 6 engineering graduates[2]). The main aims of the test were: a) to assess trainees' abilities in solving proof problems; b) to help them to become aware of their own possible difficulties with algebra in solving such problems; c) to encourage them to be self critical of their own conceptions about the teaching of algebra.

2. Theoretical framework which support our analysis of protocols

The requirement of both identifying and communicating to the trainees the criteria needed for the analysis of these protocols lead us to search for a set of theoretical instruments that would be both appropriate to the analysis of their proofs, and in tune with the view of teaching algebra that we are promoting.

The main reference in our research is the work by Arzarello *et Al.* (1994, 2001). The authors propose a model for teaching algebra as a *game of interpretation* and highlight the need for the promotion of algebra as an efficient *tool for thinking*. An awareness of the power of the algebraic language can be developed only once the student has mastered the handling of some key-aspects that arise in the development of algebraic reasoning. In particular, the authors highlight the use of *conceptual frames* defined as an *"organized set of notions, which suggests how to reason, manipulate formulas, anticipate results while*

[2] These trainees represent a good sample of those graduates who got involved in our postgraduate courses for secondary school pre- and in-service novice teachers in the last few years. As regards to the approach to algebra, and mathematics in general, these categories of trainees do not share a common vision: physics and engineering graduates usually display an instrumental vision of mathematics (particularly pragmatic for the latter), while mathematics graduates sometimes assume a passive attitude, in particular toward proof , since they are used to repeat rather than to produce proofs.

coping with a problem", and *changes from a frame to another* and from a knowledge domain to another as fundamental steps in the activation of the interpretative processes.

In Arzarello *et Al.'s* model, a symbolic expression can become a *thinking tool* in in two essential moments: 1. when we operate transformations on it that show properties that are not initially evident; 2. when, without appealing to transformations, we interpret the expression in a new frame.

According to the authors, a good command in symbolic manipulation is related to the quality and the quantity of anticipating thoughts which the subject is able to carry out in relation to the effects produced by a certain syntactic transformation on the initial form of the expression. Boero (2001) also argues that *anticipation* is a key-element in producing thought through processes of transformation. Boero defines anticipating as "*imagining the consequences of some choices operated on algebraic expressions and/or on the variables, and/or through the formalization process"*. In order to operate an efficient transformation, the subject needs to be able to foresee some aspects of the final shape of the object to be transformed in relation to the target. Arzarello *et Al.* stress that the ability to produce anticipations strictly depends on changes in the frame considered in order to interpret the shape of the expression.

Another theoretical reference that we take as fundamental for analysing students' management of meaning in algebra is the concept of *representation register* proposed by Duval (2006). The author defines representation registers those semiotic systems "that permit a transformation of representations". Among them, he includes both natural and algebraic language (which he includes within notation systems). Duval asserts that a critical aspect in the development of learning in mathematics is the ability to change from one representation register to another because such a change both allows for the modification of transformations that can be applied to the object's representation, and makes other properties of the object more explicit. According to the author, real comprehension in mathematics occurs only through the coordination of at least two different representation registers. He analyzes the functions performed by different possible typologies of transformations. He distinguishes between *treatments* ("transformations of representations that happen within the same register") and *conversions* ("transformations of representation that consist of changing a register without changing the objects being denoted"), highlighting the importance of each of these typologies of transformations. He observes that: a) conversions are essential because they can lead to work in a new register where a treatment may be carried out most economically or most powerfully; b) it is the choice of treatments that makes the choice of register relevant, so treatments and conversions are strictly intertwined.

3. Research hypothesis and purposes

Our hypothesis is that the production of good proofs in elementary number theory depends upon the management of three main components:

a. the appropriate application of frames and coordination between different frames;

b. the application of appropriate anticipating thoughts;

c. the coordination between algebraic and verbal registers (on both translational and interpretative levels).

Therefore, the purpose of our research is to single out a sample of prototype-productions to serve as references for researchers and teacher trainers of proof in elementary number theory. We will make reference to these prototypes at two levels:

1) From the research point of view, we will use the sample to verify our hypothesis and to highlight that the lack or unsuccessfully application of one of these components leads to failure and/or blocks of various types.
2) From the point of view of practice, this sample will become an instrument to be used in teacher education. Researchers and trainees will discuss the trainees' attempts at analysis by reference to researcher categories. Trainees will have the opportunity to discuss not only the difficulties they encountered and why but also how to promote a more conscious application of the algebraic language both in themselves and in their students.

4. Research Methodology

Theoretical models we used (see above) helped us to identify some interpretative keys for both the analysis of protocols and their subsequent classification.

Our analysis focused on the following: 1) The conceptual frames chosen to interpret and transform algebraic expressions and the coordination between the different frames appropriate to those same expressions; 2) The application of anticipating thoughts; 3) The conversions and treatments applied and the coordination between verbal and algebraic registers.

5. The problem

The problem we posed to trainees is the following: *"Write down a two digit number. Write down the number that you get when you invert the digits. Write down the difference between the two numbers (the greater minus the lesser). Repeat this procedure with other two digit numbers. What kind of regularity can you observe? Try to prove what you state"* [3].

The regularity to be observed is that the difference between the two numbers is always a multiple of 9 where the multiple is the difference between the digits of the first number.

The proof requires the polynomial representation of each number: since a number of two digits m and n can be written as $10m+n$, where $m>n$, the difference can be represented as $10m+n-(10n+m)$. Through simple syntactical transformations it is possible to turn the initial expression into a form that makes the required property explicit: $10m+n-(10n+m)=9m-9n=9(m-n)$.

5.1 *A priori* analysis of the problem in relation to the theoretical tools of reference

The initial conceptual frames to which the statement of the problem refers are 'difference between numbers' and 'two digits numbers'. It can be assumed, therefore, that the student will not automatically choose the 'polynomial notation' frame to represent the problem and apply the necessary simple treatments to make the conjectured property explicit. We expect that some students might apply the 'positional representation of a number' frame and then get stuck. We expect that students will intuitively apply the 'divisibility' frame that allows them to foresee the desired final shape of the expression after correct treatments, i.e. $9 \cdot k$, where k is a natural number.

A crucial passage in the proof, then, is the conversion from verbal to algebraic language, made possible only by the reference to the 'polynomial notation' frame.

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[3] This situation offer an interesting starting point to work on the construction of the conjecture's statement in a relational form. In fact it requires the previous explication of the character of the considered numbers.

Possible blocks in the treatments to perform on the initially constructed polynomial expression can be ascribed to interpretative difficulties, which are strictly related to students' inability to correctly anticipate the finale shape of the considered expression (it is necessary to recognize the transformation that leads to an expression that can be easily interpreted in the final frame ‘divisibility’).

Finally, we make some observations about possible student behaviour. It appears that many students end their numerical explorations after having observed that the difference between the two numbers is always a multiple of 9, without recognizing the relationship that exists between the two digits of the first number and the difference between the two numbers. Consequently, the analysis of the final expression could provide another index of a students' interpretative abilities, in that access to the new meanings it embodies depends on those abilities.

6. Analysis and classification of students’ protocols in relation to the theoretical components of reference

This paragraph is devoted to the analysis of trainees’ proofs or attempts at proofs when solving the examined problem. The proofs we examine here are prototypes of trainee behaviours. These prototypes are presented in increasing order with respect to quality of protocols, in relation to the incidence and the interrelation between the following: a) the application of and coordination between frames, b) ability in the game of interpretation required to produce the proof, d) display of appropriate anticipating thoughts, d) ability to correctly perform treatments and conversions, e) ability to coordinate between verbal and algebraic registers. We referred to these components to produce a careful analysis of trainees’ protocols and to classify the different highlighted typologies of protocols. In the following, we propose both a step-by-step analysis (highlighting the incidence of the components of our theoretical framework) and a global analysis (stressing on the thought processes underlined) of every prototype-protocol.

6.1 Typology 1: Application of incorrect frames

The protocols belonging to this typology are those of the students who get stuck after applying an inappropriate frame (namely the frame ‘positional notation’) which inhibits further treatments.

Protocol 1, Typology 1	Incidence of the theoretical components: our analysis
1. A trainee represents a number of two digits *m* and *n* through the expression *mq* and constructs the difference in this way:	**1.** Rigidity in the coordination between the ‘positional notation’ and the ‘polynomial notation’ frames. Incorrect conversion and inability to detect the illegitimacy of the produced expression.
2. [redacted] *with* $m \neq q$ *then* $mq\text{-}qm=9n$	**2.** Lack of control of the representation: production of semantically incoherent expressions.
3. She gets stuck.	**3.** Inability to activate anticipating thoughts.

The *first protocol* represents the most widespread erroneous approach in relation to the application of the frame ‘positional notation’: the trainee represents a two digit number through the simple juxtaposition of the two letters which represent the digits.

Protocol 2, Typology 1	Incidence of the theoretical components: our analysis
1. A trainee represents the two digits numbers through the following expression: *(a,b), (b,a) such that. b=a-n* **2.** She considers the difference between the two expressions, trying to highlight the difference n between the two digits of the initial number, and writes: **3.** *(a,a-n)-(a-n,a)=9×n* She gets stuck.	**1.** Reference to relational aspects in highlighting the connection between the digits. **2.** Activation of the frame 'multiple': attempt to control the applied conversion distinguishing between the representation of two digits numbers and the representation of products. **3.** Partial application of anticipating thoughts: control of the representation of the observed property (*9×n* highlights the connection between the multiple of 9 and the digits of the initial numbers).

In the *second protocol* we can observe a minimum attempt at reasoning. In fact the trainee introduces the letter *n* trying to make the difference of the two digits of the initial number explicit (but n is too limited). Then, showing an exploratory attitude towards the problem, she tries to set up an equation, but she is unable to elaborate it.

Protocol 3, Typology 1	Incidence of the theoretical components: our analysis
A trainee represents with *da* the tens digit of the initial number and with *u* the units digit and writes: **1.** *Natural number of two digits (da,u).* And adds: *Inverting the digits:* $(u\times 10, da\times 10^{-1})$. Then she constructs the considered difference, writing: **2.** *Difference between the > and the < :* $d=(da,u)-(u\times 10, da\times 10^{-1})$ And names her conjecture: **3.** *Starting from the difference I obtain a multiple of 9 (9 number of digits).*	**1.** Inadequate conversion: coding of two digits number through the separation between the tents block and the units block. **2.** Naive conversion between spontaneous registers ('syncopated' and algebraic register). **3.** Lack of conversion of the thesis from verbal to algebraic register. Inability to apply anticipating thoughts.

The *third protocol* can be considered a failed attempt to reconcile two different frames ('polynomial notation' and 'positional notation') that are partially elaborated in the trainee's mind. It is important to highlight the trainee's serious difficulties[4] in representing the terms of the problem. In fact, the expression *(u,da)* would make us think that the trainee uses *u* and *da* to represent the initial number digits, but the subsequent representation of the second number through $(u\times 10, da\times 10^{-1})$ displays a clear incoherence in the representation adopted.

[4] The difficulties shown by the trainee become alarming if we consider that she is a mathematics graduate. Because of lack of space, we are not going to deal with the cultural aspects that could be recalled by these 'alarm bells'. We dealt with this question in a paper presented during Cerme 5 (Cusi & Malara, 2007).

6.2 Typology 2: Attempt to apply a suitable frame, but inadequate conversion

Typology 2	Incidence of the theoretical components: our analysis
1. He considers some numerical examples and writes: *"$ab \in N$, $ba \in N$, with $a \neq b$ and $ba > ab$"*, And, straight away, he comments*:* *"difficulties in the generalization. This representation doesn't seem correct to me!"*. He then expresses his conjecture: *"In case the two digits are different, the difference between the greater number and the lower is always a multiple of 9 or 9"*.	**1.** Initial activation of the frame 'positional notation' and erroneous conversion.
2. The trainee tries to formalize what he asserts, writing $ba-ab=3ab$ then he cancels and writes $ba-ab=9n$ and again *"no!"*	**2.** Lack of control of the coherence of the representations and of the coordination of the activated frames.
3. After getting stuck, he tries to understand how to formalize the problem and writes: [redacted] then [redacted]	**3.** Change of frame: attempt to refer to the 'polynomial notation' frame. Correct treatments in the numerical-symbolic register: coordination between 'positional notation' and 'polynomial notation' frames in particular numerical examples.
4.He tries to generalize and writes: [redacted]	**4.** Erroneous extension of the treatments to the general situation: inability to coordinate 'positional notation' and 'polynomial notation' frames.
5. And then*:* $(ha+kb)-(ka+hb)=9n$ $(ha+kb)-ka-hb=9n$ $h(a-b)+k(b-a)=9n$ But he writes *"I get stuck. I can't go on"*.	**5.** Impossibility to perform anticipating thoughts: application of unproductive treatments.

The trainee who produces this protocol, probably because of gaps in his basic knowledge, initially relies on the frame 'positional notation', but then sees that this choice is not productive and indicates so. His comments, which appear all through his protocol, display his awareness of the limits he has in using algebraic language. He perceives that, in order to prove his conjecture, he needs to perform transformations that lead to an expression of the form $9n$, where n is a natural number[5].

Subsequently, the trainee starts searching for an appropriate frame from which to carry out the conversion from the verbal to the algebraic register and to construct an expression to be productively manipulated. In order to construct such an expression he considers some numerical examples as starting points for a generalization, but his conversion is not-controlled: it can be defined a sort of 'over-generalization' (the trainee represents a number in polynomial notation assigning to the constants, that is to the powers of ten involved, a variable value). This conversion to an 'over-generalized' form results in him being unable to carry out productive treatments, so any possible anticipation is blocked before it arises.

[5] We can however observe that his conception of the multiple of a number is incomplete, since he does not consider 9 a multiple of itself.

6.3 Typology 3: Blind reference to algebraic register: 'good' application of frame, but not related to (a control of) the sense and the aims of the produced expressions

Typology 3	Incidence of the theoretical components: our analysis
1. The trainee writes: *n two digit number, a=tens number, b=units number* *n=a·10+b so n'= b·10+a is the number I obtain inverting tens and unit.* **2.** *I suppose a>b. It follows that n>n'.* *I consider the difference:* **3.** *(a·10+b)-(b·10+a)=(a-b)·10+(b-a)* **4.** *This expression represents the way in which the difference is constructed starting form the digits.*	**1.** Activation of the 'polynomial notation' frame and correct conversion between verbal and algebraic register. **2.** Correct interpretation and good control of the expressions in the activated frame. **3.** Correct treatments in the algebraic register. **4.** Erroneous interpretation of the final expression in the frame 'polynomial notation'. Lack of anticipating thoughts.

This protocol is particularly problematic because the trainee does not feel the need to explore the situation through numerical examples and does not formulate a conjecture. Without any doubts she singles out, the 'polynomial notation' frame and shows that in this frame she is able to handle and control the concept of 'being greater than'. However, she relies upon algebraic language 'blindly'. After having correctly represented the difference between the two numbers, she does not further an analysis of the expression she obtained through syntactical transformations. Her block is primarily interpretative: she is able neither to interpret the final constructed expression nor to evaluate whether or not it is the polynomial representation of a two digit number (it is not because b<a) as a result of not invoking the frame 'directed integers'. By not formulating a conjecture she does not state her objectives and so does not control what she has to prove.

In our opinion this protocol is a clear example of the strict 'bidirectional' relationship between interpreting processes and frame application. Frame application is a necessary condition for the development of the interpreting processes. Correspondingly, we can only assess the adequacy of a fixed frame with respect to a produced expression when we enter the interpreting processes. If we find it to be inadequate we can then try a new frame. In this protocol, for example, the trainee does not try to analyse the expression *(a-b)·10+(b-a)* in the 'polynomial notation' frame. This voluntary interpreting block constrains the application of the frame 'directed integers', which is needed to see *a-b* and *b-a* as opposite numbers and to carry out further treatments on the examined expression.

6.4 Typology 4: Appropriate application of frame, but not supported by semantic control and anticipating thoughts

Typology 4	Incidence of the theoretical components
The trainee constructs a table to collect numerical examples. Then she proposes this conjecture: *If the difference between the two digits is 1, the difference between the two numbers is 9; if the difference between the digits is 2, the difference between the numbers is 18, etc...*	
1. She correctly represents the problem: $\left[(x \cdot 10)+y\right]-\left[(y \cdot 10)+x\right]$	**1.** Activation of the 'polynomial notation' frame and correct conversion between verbal and algebraic register.
2. [redacted] and gets stuck.	**2.** Application of anticipating thoughts and 'improvement' of the produced representation. Application of a correct treatment. Lack of anticipating thoughts in relation to the objective.

This protocol reveals an instance of the fundamental role played by semantic control in the anticipating processes. The trainee invokes the frame 'polynomial notation', but is not able to carry out the transformations that would lead her to the expression which makes the observed property explicit.

We notice that the trainee correctly represents the initial difference, even making explicit the relationship between the two digits in relation to the difference between them. This representative choice is strictly related to the fact that she realises that the difference between the two numbers is a multiple of 9 and that the multiple is the difference between the two digits. Her intuition, however, is not supported by an anticipating thought that foresees that the expression she needs to obtain is *9k* (where *k* is a natural number): the manipulations she carries out are 'blind' and they do not relate to the interpretation of the final expression.

6.5 Typology 5: Appropriate application of frames and the game of interpretation coherent with the objective. Ability to 'read' through the symbolic expression properties not observed during the exploratory phase.

Typology 5	Incidence of the theoretical components: our analysis
The trainee considers two numerical examples: *21-12=9 32-23=9* Then writes: *The difference is always 9.*	
1. And adds: *I consider* $a,b \in N$ / $a,b<10$ *with* $a<b$ *First number:* $a\cdot10+b$ *Second number:* $b\cdot10+a$	**1.** Activation of the 'polynomial notation' frame. Correct conversion between verbal and algebraic register. Good control of the meaning of the produced expressions.

2. She constructs the expression-difference between the two numbers and she transforms it: *$b\cdot 10+a-a\cdot 10-b=(b-a)\cdot 10-(b-a)=(b-a)(10-1)=9(b-a)$* commenting:	**2.** Application of anticipating thoughts which guide a correct chain of treatments.
3. *b-a always positive ⇒ The difference between the two numbers is always a multiple of 9 (In my examples I always considered b-a=1).*	**3.** Interpretation of the final expression through a good coordination between the 'multiple' frame and the 'positional notation' frame.

The trainee initially works on numerical examples and formulates a partial conjecture. She invokes the 'polynomial notation' frame and correctly handles the concept 'being greater than' in this frame. She also operates a good conversion from verbal to algebraic language. The manipulations she carries out are prefaced by an anticipating thought appropriate to her objective. We notice that the trainee displays good interpretative abilities. She reanalyses the final expression by reference to the examples she started with. In doing so, she realizes that in her examples she only considered numbers whose digits differ by 1, and she proposes an expansion of her initial conjecture. However, she does not 'read' all the information the expression could provide, possibly limited by the formulation of her initial conjecture .

6.6 Typology 6: Good coordination between frames and good interpretation of the expressions in the applied frames

Typology 6	Incidence of the theoretical components: our analysis
1. The trainee represents the two numbers in this way: $n_1=c_2\cdot 10+c_1$ $n_2=c_1\cdot 10+c_2$ $c_2>c_1$ Then writes: $n_1-n_2=(c_2-c_1)\cdot 10+(c_1-c_2)$	**1.** Activation of the polynomial notation frame. Correct conversion between verbal and algebraic register. Application of correct treatments.
2. Commenting: *$c_2-c_1>0$ ok!* *$c_1-c_2<0$*	**2.** Good control of the obtained expression in relation to the 'polynomial notation' frame.
3. *I need a 'loan'* *so $n_1-n_2=(c_2-c_1-1)\cdot 10+(10+c_1-c_2)$*	**3.** Activation of the 'subtraction's algorithm' frame and application of a correct treatment in the new frame.
4. He finally concludes: *I consider the sum between the digits of n_1-n_2 :* *$c_2-c_1-1+10+c_1-c_2=9$.*	**4.** Activation of the 'divisibility' frame and coordination between 'positional notation' and 'polynomial notation' frame. This approach is the result of a previous anticipating thought, related to a good coordination between the 'divisibility' and the 'positional notation' frame.

The trainee correctly invokes the 'polynomial notation' frame to represent two digit numbers and invert the order of the digits, and displays ability in handling the concept 'being greater than' in this frame.

After having correctly constructed the expression which represents the difference between the two numbers and having carried out syntactical transformations on it to take it back to something similar to its initial shape, he displays a good command of the meaning of such expressions in the 'polynomial notation' frame by noting that the obtained expression is not the polynomial representation of the difference n_1-n_2.

In order to represent the results in polynomial notation, he invokes two further frames: the 'positional notation' and 'subtraction algorithm' frames. He appears to avoid an analysis of the final expression in the 'being multiple' frame because of a sense that he should proceed in the same ‘polynomial notation’ frame and with a numerical field of reference (natural numbers).

Though his choices are not the most efficient, this protocol displays an excellent ability in the management of multiple frames and remarkable abilities in the semantic command of the managed expressions.

7. Conclusions

The analysis and the classification we propose, even if still in progress, allow us to offer some initial conclusions with respect to some of the aspects we want to discuss in this final section: 1) Effectiveness of the theoretical references we selected as tools for analyzing student proofs in elementary number theory; 2) Essential components for good productions in this context; 3) The role played, in the protocols we analyzed, by the three component we singled out (application and coordination between frames, anticipating thoughts, coordination between the verbal and algebraic registers) and the mutual relationships between them.

Referring to point 1, we can observe that the theoretical references we used in this work as tools for our inquiry turned out to be valid both for analyzing and classifying trainee protocols. In the future development of this research we will test this model for possible influences that may result from communication between peers by analysing student protocols when they work in pairs and in small groups.

The final protocol we analyzed provides a clear answer to the question posed in point 2: it demonstrates that the appropriate application and combination of the three components mentioned above is a necessary and sufficient condition for the proper development of a proof in elementary number theory.. That is: appropriate application of frames and appropriate coordination between them; appropriate anticipating thoughts; appropriate coordination between algebraic and verbal registers.

Through our analysis we were able to both observe some of the effects of the lack or otherwise of one or more of the 'components' and focus on the game of mutual relationships between them. Protocols 1, for example, testify how the non-application of the correct initial frame, almost always as a result of a lack of basic knowledge of elementary representations, necessary leads to irremediable blocks in the development of the proof. Lack of knowledge also accounts for blocks in the conversion from the verbal to the algebraic register which follows the identification of the initial frame. In protocol 2, for example, the trainee invokes the correct frame but is unable to apply it fully and as a result converts incorrectly, which leads to his block. This fact testifies that the application of the correct initial frame is not a sufficient condition for the completion of the proof if it is not well supported by flexibility in carrying out conversions from the verbal to the algebraic register.

Besides testifying to the serious consequences of having ‘blind’ confidence in the algebraic language, protocol 3 also provides a clear example of the presence of a mutual and strict relationship between the interpretative processes and the application and coordination of between frames. Success in the interpretative processes depends on the frame invoked, but, simultaneously, the application of new frames is only possible following an ‘attempt’ to analyse the expressions as they are constructed. When manipulation is ‘blind’ the interpretative processes are blocked before they arise.

The fundamental role played by anticipating thoughts, as a ‘guide’ for the transformations to be carried out, is highlighted in protocol 4, which reveals how an inability to foresee the expression to be attained blocks both treatments and the interpretative processes them selves.

Finally, protocol 5 testifies to the effectiveness of the interpretative process when supported by appropriate coordination between frames: even though the partial conjecture proposed by the trainee limits her anticipating thoughts, the need for further analysis of the expressions as they are arise induces her to recognize and overcome the limits imposed by her initial approach to the problem.

We want to conclude this paragraph with a reflection on the importance of our work from the point of view of the connection between theory and practice. The use of this type of analysis in activities with trainees is important for their formation in that it helps them to become more aware not only of their own errors and limitations in the use of the algebraic language, but also of the kind of models it is necessary for them to adopt in the presentation of proof problems in elementary number theory.

REFERENCES

- Arcavi, A.: 1994, 'Symbol sense: informal sense-making in formal mathematics', *For the Learning of Mathematics* 14(3), 24–35.
- Arcavi, A.: 2005, 'Developing and using symbol sense in mathematics', *For the Learning of Mathematics* 25(2), 42-47.
- Arzarello, F., Bazzini, L., Chiappini, G.: 1994, 'L'algebra come strumento di pensiero. Analisi teorica e considerazioni didattiche'; Progetto strategico TID, Quad.6, Collana Innovazioni didattiche per la matematica.
- Arzarello, F., Bazzini, L., Chiappini, G.: 2001, 'A model for analyzing algebraic thinking', in Sutherland R. & Al. (eds.), *Perspectives on School Algebra*, pp. 61-81. Kluwer (The Netherlands).
- Bell, A., 1996: 'Problem solving approaches to algebra: two aspects', in Bednarz and al. (eds.) *Approaches to algebra (Perspectives for research and teaching).* Kluwer (The Netherlands).
- Boero, P, Garuti, R., Lemut, E., Mariotti, M.A.: 1996, 'Challenging the Traditional school approach to theorems: a hypothesis about the cognitive unity of theorems', *Proc. PME 20*, University of Valencia, vol. 2, 113-120.
- Boero, P.: 2001, 'Transformation and Anticipation as Key Processes in Algebraic Problem Solving', in R. Sutherland & Al. (eds.), *Perspectives on School Algebra*, pp. 99-119. Kluwer (The Netherlands).
- Brown, L., Drouhard, J.P., 2004: 'Responses to 'The Core of Algebra''. In Stecey, K. and al., *The future of the teaching and learning of algebra. The 12th ICMI study*, 35-44. Kluwer (The Netherlands).
- Conference Board of the Mathematical Sciences, 2001: *The Mathematical Education of Teachers*. Providence RI and Washington DC: American Mathematical Society and Mathematical Association of America.
- Cusi, A., Malara, N.A.: 2007, 'Proof of statements in elementary number theory: analysis of trainee teachers productions', in Pitta-Pantazi, D. & Philippou, G., *Proceedings of the Fifth Congress of the European Society for Research in Mathematics Education*, pp. 591-600. Erme & University of Cyprus (Larnaca).
- Duval, R.: 2006, 'A cognitive analysis of problems of comprehension in a learning of mathematics', *Educational Studies in Mathematics* 61, 103–131.
- Kieran, C., 2004: 'The core of algebra: reflections on its main activities'. In Stecey, K. and al., *The future of the teaching and learning of algebra. The 12th ICMI study*, 21-34. Kluwer (The Netherlands).
- Kieran, C., 2006: 'Research on the learning and teaching of algebra', *Handbook of Research on the Psychology of Mathematics Education*, Gutierrez, A. & Boero, P. (eds.), Sense Publishers (Rotterdam).
- Malara, N.A., 2001: 'La dimostrazione in ambito aritmetico, quale spazio nella scuola secondaria?', in Malara N.A (ed.), *Educazione Matematica e Sviluppo Sociale: esperienze nel mondo e prospettive,* Rubettino (Soveria Mannelli), 129-166.
- Malara N.A., 2003: 'Opinioni sull'algebra di futuri insegnanti: incidenza del retroterra culturale', *La matematica e la sua didattica,* 1, 17-42.
- Selden, A. & Selden, J., 2002: 'Reflections on Mathematics Educational Research Question in Elementary Number Theory', in Campbell, S. R., Zazkis, R., *Learning and teaching Number Theory*',

Ablex Publishing(Westport), 213-230.
- Wheeler, D., 1996: ‘Backwards and Forwards: Reflections on different approaches to algebra’ in Bednarz and al. (eds.), *Approaches to algebra (Perspectives for research and teaching)*. Kluwer (The Netherlands).
- Zazkis, R. and Campbell, S. R. (2006). Number Theory in Mathematics Education: Perspectives and Prospects. In Campbell, S. R., Zazkis, R. (eds.), *Number Theory in Mathematics Education: Perspectives and Prospects* (pp.1-18). Routledge (London).