THE TEACHER'S ROLE IN DEVELOPING PROOF ACTIVITIES THROUGH ALGEBRAIC LANGUAGE

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In this paper, part of a wider study aimed at a conscious use of the algebraic language in proof activities in elementary number theory (ENT), *we analyse the role of the teacher in discussion activities aimed at providing pupils with the activation of those skills needed for thinking through algebraic language, and we will highlight the specific characteristics of the teacher's effective action, within this process, in providing a role model to the pupils.*

THEORETICAL FRAMEWORK

The work presented herein is part of a wide study (Cusi 2009a), within the Italian framework of research for innovation (Arzarello & Bartolini Bussi 1998). It is based on the need to promote, in the actual school environment, activities aimed at helping the pupils in developing a symbol sense (Arcavi 1994) towards a conscious use of algebraic language in producing thought (Arzarello *et al.* 2001), possibly re-converting their previous pseudo-structural view of algebra (Sfard & Linchevski 1994). This led to designing and testing a novel path of introduction to proofs in ENT (grades 9-10), with a view to an approach in teaching/learning algebra focussed on the control of meanings (Cusi 2009b). The theoretical background of the entire research effort is the Vygotskyan framework of approach to teaching and learning processes, specifically in reference to: (a) Vygotsky's (1978) ideas on the central role of social interaction in thought-development processes and on the importance of contribution of adults (or more skilled classmates) in helping to bridge the gap between potential and actual development; and (b) Leont'ev's (1977) ideas on the role played by activities performed in a social context in the development of the individual, of the personal meaning that he or she attaches to those activities and of the stimuli and motives that determine in him or her a true awareness of learning processes. In this framework, the role played by the teacher is fundamental. Such role has been and is the subject of several studies, especially as far as group activities in the classroom are concerned. From the viewpoint of the social aspects of interaction, for instance, some have underlined how the teacher should be able to create a good interaction context, acting as a true participant, stimulating and regulating the argumentative processes (Schwarz *et al*. 2004) and focussing the pupils' attention on the need to check the significance of other people's interventions in order to offer any criticism or suggestions (Wood 1999). As far as the mathematical knowledge to be discussed, the teacher has the responsibility of directing its development, "filtering" the students' ideas in order to focus the attention on those aspects which hold a particular relevance and significance (Sherin 2002). The teacher is responsible of the discussion's quality because, through his or her reaction to the pupils' interventions, he or she implicitly evaluates the solution they propose (Yackel & Cobb 1996), leading them to acquire an awareness of which forms of reasoning are more sophisticated (Anghileri 2006). Several studies have highlighted the methodologies and behaviours that a teacher should follow in order to manage and develop interactions in a flexible and dynamic fashion (Anghileri 2006, Leikin & Dinur 2007).

In the case of mathematical proof, the role played by the teacher during mathematical discussions becomes even more complex and delicate. Balacheff (1991) observes how, from time to time, interaction might even pose an obstacle to the learning of proof processes since it favours the development of those "argumentative behaviours" which work against attaining an awareness of the specific nature of mathematical proof and of the role of logical deduction. More recently, Martin *et al*. (2005) have highlighted the need to focus the attention on the role played by the teacher in developing the learning of mathematical proof, underlining the impact of his or her choices and actions both on individual and collective comprehension. They also stress the fact that the teacher should facilitate the participation of students in the process of collectively negotiating the conjecture and collectively constructing the proof, teaching them how the development of mathematical discourse must follow a number of well-defined rules. Blanton *et al.* (2003), in a study on university students, explored the value of different scaffolding practices in the development of demonstrative processes, stressing the importance of negotiation processes and highlighting the value of "metacognitive acts" in group discussions.

These aspects concern, even more so, the construction of proofs through algebraic language, since it requires that the students employ skills that may be attained only when they participate in contexts, such as those advocated by Schoenfeld (1992), wherein mathematical sense-making is practiced and developed.

Previously (Cusi 2009b), through the analysis of discussions held by groups of students working on the construction of proofs through algebraic language, we highlighted those skills that must be activated in the students so that they can be able to succeed in this kind of activity and acquire an awareness of the role played by algebraic language within these processes. In order to analyse the development of thought processes through the use of algebraic language we referenced concepts drawn from the works of Arzarello *et al*. (cit), Boero (2001) and Duval (2006): the first authors highlighted the importance of developing *conceptual frames* (defined as "organized set of notions, which suggests how to reason, manipulate formulas, anticipate results while coping with a problem") and being able to switch frames in order to interpret the algebraic annotations being progressively constructed; the second work is focussed on the concept of *anticipation* and its role in thought production through transformation processes (Boero defines anticipating as "imagining the consequences of some choices operated on algebraic expressions and/or on the variables, and/or through the formalization process"); the third author found a key aspect in the general learning of mathematics in the *coordination* *between different representation registers* (defined as those semiotic systems "that permit a transformation of representations"). Our analysis allowed us to highlight the key role played, in the development of proof activities, by three essential components and their mutual relationships: (1) Appropriate activation of frames and good coordination between different frames; (2) Correct enacting of anticipatory thought; and (3) Good flexibility in coordinating algebraic register and verbal register. These studies increased our awareness of the key role played by the teacher in leading the pupils to the attainment of these skills; this led us to study in depth the analysis of the teacher's actions in the group discussions introducing proof through algebraic language. This paper shows the main results of this part of our research.

HYPOTHESES AND OBJECTIVES OF THIS STUDY

The idea this part of the worked is based on is that, in classroom interaction, pupils acquire, through a process of cognitive apprenticeship (Collins *et al.* 1989), the teachers' stances and behaviours. This led us to state the following two hypotheses: **(a)** The teacher's demeanour and actions must be tailored to facilitate, in the students, an actual building of the three skills we mentioned before, which are they need in order to face and comprehend proofs through algebraic language; **(b)** The teacher has the critical task of being a *role model* in leading the students to acquire, gradually and consciously, these skills.

The research objectives related to these hypotheses are: **(1)** To study the teachers' behaviour in leading the students to the construction of proofs through algebraic language and to highlight **(a)** the productivity or negativity of their interventions, pointing out which behaviours characterise a conscious and effective teaching; **(b)** the effects on the students of the approach they follow, both in acquired skills and in demonstrated awareness. **(2)** To identify the specific characteristics of a teacher which effectively act as a role model for the pupils and to give a first characterisation to the theoretical construct of the *teacher as a role model*.

RESEARCH METHODOLOGY

The classroom work being studied has proceeded through group discussions, smaller-group work and individual tests. The data being analysed consists of the students' written tests and the transcripts of audio recordings of classroom discussions and smaller-group activities. Each transcript has been analysed from different viewpoints according to the type of activity. For group discussions, the analysis has been performed by highlighting: (1) weak and strong points in the discussions concerning the three key components in developing arithmetical proofs; (2) the role played by the teacher as a "stimulus" for the activation of attitudes of approach to algebra as an instrument of thought, both as "role model" and "leader" in the production of reasoning. The transcripts of smaller-group activities have been analysed with the aim to highlight: (a) weak and strong points in the discussions concerning the three key components in developing arithmetical proofs; (b) influences of the approach followed by the teacher on the choices made by the students, with special regard to "meta"-type considerations within the groups. Space limitations do not allow us to dwell on analysis of smaller-group activities; therefore, we will only report the analysis of a discussion led by an effective teacher.

EXAMPLE OF A TEACHER ACTING AS AN EFFECTIVE ROLE MODEL

The following excerpt is in reference to a discussion about the proof of the following proposition: *If b is an odd number, the expression 3b represents an odd number*. It should be noted that, from the very beginning of the discussion, the teacher chose to adopt an algebraic approach to demonstrating propositions, however simple, as in this case, with the aim to facilitate the students' transition from verbal argumentation to algebraic proof. We propose this protocol because it shows in detail both the methodology adopted by the teacher (T) to facilitate the building of the skills we identified as essential for producing thought through algebraic language, and the favourable effects of this methodology on the pupils.

In the first part of the discussions, all pupils agree in formalising the hypothesis through the equality $b=2x+1$ and spontaneously suggest substituting $2x+1$ for b in the expression $3b$. T, then, writes on the blackboard: $3b=3(2x+1)$. This excerpt is from the following part of the discussion.

1 T: What can I do then? (*Silence*) How do you bring out the fact that a number is odd?

2 M: You take a multiple of 2 and add it to 1…

3 T: Does this look like a multiple of 2 added to 1?

4 A: Yes, and then it's multiplied by 3!

5 T: Well... all I have now is 3 times something… you said that what I want is a multiple of 2 plus 1! Does this look like a multiple of 2 plus 1?

6 A: You can write 3 times $2x$, in parentheses, plus 1! [T *writes* $3(2x+1)=3\cdot(2x)+1$]

7 T: Are these two expressions equivalent?

8 (Everyone): No!

9 T: Watch out, then, because I must use transformations which give me an expression equivalent to the one I started from!

10 S: You can write 3 times, open bracket, $2x$ in parentheses, plus 1.
[T *writes* $3(2x+1)= 3[(2x)+1]$]

11 M: Why'd you do that? It's the same thing!

12 T: Let's keep our target in mind... What I want to do is to bring out a +1, but in the main expression. We have brought out a +1, here, but it's inside a factor…

13 A_N: Since we can calculate, we could make that $6x+3$…

[T *writes* $3(2x+1)=6x+3$]…

Then it becomes $6x+2$, in parentheses, then +1.

[T *writes* $3(2x+1)=6x+3=(6x+2)+1$]

14 T: She said, "If I calculate, I get $6x+3$". What is her aim, then? To show that we have a +1. Can we see, now, how it's an even number plus 1?

15 (Everyone): Yes!

16 T: Couldn't we do something to bring it out even more?…

17 M: We can factorise the expression in parentheses! Yes! We can gather a 2 there!

18 T: Let's do that… We have 2 times $3x+1$, plus 1.
[T *writes* $3b=3(2x+1)=6x+3=(6x+2)+1=2(3x+1)+1$]

19 T: Look, now. We've started from $3b$ and we got 2 times something plus 1. Like M said, an even number plus 1 always gives…

20 S: An odd number.

21 T: Then our choice is justified, rigorously! Can you follow the reasoning?

22 (Everyone): Yes!

Analysis of the excerpt

T asks the class how to proceed (line 1), assuming the role of *investigating subject* and part of the class group in the "research" work being activated. Since the class remains silent, T asks the students whether this expression [$3(2x+1)$] explicitly show its nature as an odd number (line 1). Thus, she becomes an *activator of anticipatory thought*. Note that the teacher's aim is to lead the pupils to syntactically transform the expression they constructed so that its representation of an odd number is evidenced, without relying on any verbal argument. That is why T *stimulates* the pupils through a more explicit request about the final form they should obtain (line 3). For the difficulties encountered by A (line 4) in following the reasoning, T elects to *provoke* both a correct interpretation of the expression being studied and an effective approach to the manipulations the pupils should enact, making the target once again explicit (line 5). The class seems to have followed the teacher's indications, but has difficulties in understanding which procedures to apply to the expression in order to better highlight the fact that it represents an odd number. The *anticipatory thought* that T tried to activate in the pupils ("I need to bring out the sum of an even number and 1") *predominates on the control of manipulations* enacted by the pupils, *leading them erroneously* [$3(2x+1)=3\cdot(2x)+1$, line 6]. Having seen the wrong treatment proposed by A, T writes on the blackboard the equality he suggested, points out how the two expressions are not equivalent and puts herself *on a meta level* commenting on the meaning of the activity as a whole (the transformations to be used on the expression being studied must be 'legitimate', i.e. they must lead to expressions actually equivalent to the starting one, lines 7 and 9). Here, the teacher acts as a *stimulator of thoughtful attitudes* about the meaning of the activity. T then lets herself be led by S (line 10), who seems to have lost sight of the target: his *anticipatory thought* is inhibited *by his need to verify the equivalence between the expressions*. Therefore, T's task becomes *to facilitate* in the pupils *a balance between semantic aspects* (interpretation of expressions in the activated conceptual frames and anticipatory thought regarding the activity's target) *and syntactic aspects* (controlling the correctness of the manipulations used). M, who intervenes offering a meta comment on the meaning of the activity (line 11), shows he has received the teacher's stimuli: his exclamation, "It's the same thing," is not about the fact that the two expressions are equivalent to each other, but that they both activate the same (multiple) frame, wherein they have the same interpretation. To try and lead the class in the right direction, T puts herself on the meta level once more and recalls the aim of the manipulations being performed (line 12), assuming the role of *activator of anticipatory thought*. An indication of this characteristic role of T's is the frequency with which she uses the word "target". At this point, A_N follows T's lead and suggests transforming the expression into *additive form* (line 13), analyses it in a new frame (even-odd), acting on it in order to make the +1 addend explicit. In this way she shows a good degree of semantic control over the management of the manipulations she applies under the guidance of a correct anticipatory thought. T (line 14) goes back to A_N's line of reasoning (*re-phrasing*, in the words of Anghileri, 2006) to stimulate a moment of thought on the effectiveness of her approach. This highlights another of the teacher's roles, that of *leading the class in recognising effective reference models*. When T asks (line 16) to suggest a treatment that will make the nature of $(6x+2)+1$ as an odd number more explicit, M (line 17) interprets T's question in the 'even-odd' frame and suggest the correct treatment. T closes the discussion (lines 19 and 21), putting herself once more on the meta level and reviewing the entire process. The thoughts on the meaning of the transformations performed and on the validity of the expression obtained, together with the continuing stimuli throughout the discussion, are aimed at *inducing a meta-level attitude* also in the pupils.

The analysis of smaller-group discussion protocols, not documented herein, allowed us to show how the student successfully adopted the teacher's approach as a reference model to face the subsequent learning activities. Such protocols are rich in demonstrations of acquired skills and awareness.

CONCLUSION

The analysis of this and other discussions led by the teachers we studied and of the group activities by the same pupils allowed us to identify the *specific characters of the effective action of a teacher in becoming a role model for the students*, thereby identifying the profile of an 'effective teacher'. This led us to the idea of defining the theoretical construct of the '*teacher as role model'*. The defining elements of that construct are as follows: the effective teacher must (a) be able to assume the role of "*investigating subject*", stimulating an attitude of research on the problem being studied, and of an *integral element* of the class group in the research being activated; (b) be able to assume the role of *operational/strategic leader*, through an *attitude towards sharing* (as opposed to transmission) of knowledge, and as a *thoughtfulness leader* in identifying efficient operational/strategic models during class activities; (c) be aware of his or her responsibility in maintaining a *harmonized balance between semantic and syntactic aspects* during the collective production of thought through algebraic language; (d) seek to *stimulate and provoke* the *building of key skills* in the production of thought through algebraic language (be able to translate, interpret, anticipate, manipulate), acting as an *"activator" of interpretation processes* and *"activator" of anticipatory thought*; (e) also have the aim to stimulate and provoke

meta-level attitudes, acting as an *"activator" of thoughtful attitudes* and *"activator" of meta-cognitive acts*. We tested this model in the analysis of other discussions pertaining to the whole path being studied: this allowed us to highlight a clear distinction between teachers who are able to act as role models to the class and teachers who are not able to interact effectively with the class; the latter, in the most extreme cases, produce the opposite effect: stimulating some sort of pseudo-structural approach to the use of algebraic language in developing thought processes.
In the future, our research will be aimed at verifying the effectiveness of the construct we proposed, both in proof-related activities outside the field of arithmetic and in algebraic activities involving logical, interpretation and coordination aspects between different representation registers.

References

Anghileri, J. (2006). Scaffolding practices that enhance mathematics learning. *Journal of Mathematics Teacher Education,* 9, 33–52.

Arcavi, A. (1994). Symbol sense: informal sense-making in formal mathematics. *For the Learning of Mathematics*, 14(3), 24–35.

Arzarello, F. & Bartolini Bussi, M.G. (1998). Italian Trends of Research in Mathematics Education: a National Case Study in the International Perspective. In J. Kilpatrick & A. Sierpinska (Eds.), *Mathematics Education as a Research Domain: A Search for Identity* (pp. 243-262). Kluwer: The Netherlands.

Arzarello, F., Bazzini, L., & Chiappini, G. (2001). A model for analyzing algebraic thinking. In R. Sutherland R. *et Al*. (Eds.), *Perspectives on School Algebra* (pp. 61-81). Kluwer: The Netherlands.

Balacheff, N. (1991). The benefits and limits of social interaction: the case of mathematical proof. In A. Bishop *et Al*. (Eds.), *Mathematical knowledge: its growth through teaching* (pp. 175-192). Kluwer: The Netherlands.

Blanton, M.L., Stylianou, D.A. & David, M.M. (2003). The nature of scaffolding in undergraduate students' transition to mathematical proof. In N. Pateman, *et Al.* (Eds.), *Proc. of PME* 27 (vol. 2, pp. 113-120). Honolulu.

Boero, P. (2001). Transformation and Anticipation as Key Processes in Algebraic Problem Solving. In R. Sutherland *et Al*. (Eds.), *Perspectives on School Algebra* (pp. 99-119). Kluwer: The Netherlands.

Collins, A., Brown, J.S. & Newman, S.E. (1989). Cognitive Apprenticeship: Teaching the Crafts of Reading, Writing and Mathematics! In L.B. Resnick (Ed.), *Knowing, Learning, and Instruction*: *Essays in Honor of Robert Glaser* (pp. 453-494). Hillsdale, NJ: Lawrence Erlbaum Associates.

Cusi, A. (2009a). *Problematiche relative all'insegnamento/apprendimento dell'algebra e ruolo del linguaggio algebrico nell'approccio alla dimostrazione in ambito aritmetico: competenze/consapevolezze dell'allievo ed azione dell'insegnante*. Unpublished PhD dissertation: Università degli studi di Modena e Reggio Emilia.

Cusi, A. (2009b). Interrelation between anticipating thought and interpretative aspects in the use of algebraic language for the construction of proofs in elementary number theory. *Proc. CERME 6*. Lyon. (to appear)

Duval, R. (2006). A cognitive analysis of problems of comprehension in a learning of mathematics. *Educational Studies in Mathematics*, 61, 103–131.

Leikin, R. & Dinur, S. (2007). Teacher flexibility in mathematical discussion. *Journal of Mathematical Behavior*, 26, 328–347.

Leont'ev, A.N. (1978). *Activity, Consciousness and Personality*. Englewood Cliffs: Prentice-Hall.

Martin, T.S., Soucy McCrone, S.M. Wallace Bower, M.L. & Dindyal, J. (2005). The interplay of teacher and student actions in the teaching and learning of geometric proof. *Educational Studies in Mathematics*, 60, 95–124.

Schoenfeld, A.H. (1992). Learning to think mathematically: Problem solving, metacognition, and sense-making in mathematics. In D. Grouws (Ed.), *Handbook for Research on Mathematics Teaching and Learning* (pp. 334-370). New York: MacMillan.

Schwarz, B.B., Hershkowitz, R. & Azmon, S. (2006). The role of the teacher in turning claims to arguments. In J. Novotná *et Al*. (Eds.), *Proc. of PME 30* (vol. 5, pp. 65-72). Prague.

Sfard, A. & Linchevski, L. (1994). The gains and the pitfalls of reification. The case of algebra. *Educational Studies in Mathematics*, 26(2-3), 191-228.

Sherin, M.G. (2002). A balancing act: developing a discourse community in a mathematics classroom. *Journal of Mathematics Teacher Education*, 5, 205–233.

Skemp, R.R. (1976). Relational Understanding and Instrumental Understanding. *Mathematics Teaching*, 77, 20–26.

Vygotsky, L. S. (1978). *Mind in society: The development of higher mental processes*. Cambridge, MA: Harvard University Press.