METHODS AND TOOLS TO PROMOTE IN TEACHERS A SOCIO-CONSTRUCTIVE APPROACH TO MATHEMATICS TEACHING

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This work, after a short introduction on the role of the teacher in socio-constructive teaching and a synthesis of the main directions suggested by research in this field, deals with the methodology used to work with and for in-service teachers involved in a teacher training programme, aiming at a refinement of their skills in orchestrating mathematical discussions. Particular attention is paid to the description of a specific tool for teacher education we constructed starting from teachers' transcripts of their own classroom processes and going through multiple analytical comments. This tool stresses and magnifies teachers' attitudes, conceptions and knowledge as emerging from the process: the teacher is thus led to critically reflect upon his/her own action in the classroom and possibly adjust it. In this work, discussion focuses on the main problems concerning teacher education, and highlighted by the use of this tools, and ends up with reflections about its formative efficacy.

KEYWORDS: SOCIO-CONSTRUCTIVE TEACHING, TEACHER ROLE, ANALYSIS OF TEACHING AND LEARNING PROCESSES, REFLECTIONS IN CO-PARTNERSHIP, RECONSTRUCTION OF KNOWLEDGE, TOOLS FOR TEACHER EDUCATION.

INTRODUCTION

The dizzy increase in scientific and technological research led to a need for a different mathematical education, pointing to modelling and problem solving rather than to the learning of mathematical facts, and open to argumentation and cooperative work in the classroom. As a consequence, conceptions about the mathematics to be taught and the related teaching modalities have changed, as shown by the USA Standards (2000), by the International P.I.S.A. assessment test (2000/03) and also by the recent proposals made by MIUR-UMI for mathematics teaching (Anichini & Al., 2001-2004).

Current research in Mathematics Education points to *socio-constructive* teaching as the most suitable to raise pupils' interest in mathematics and help them develop a conception of the discipline, closer to real life and fitting with the current educational needs. This is particularly true for compulsory school, where a conception about the discipline is initially shaped. In this type of teaching, pupils are supposed to explore situations purposefully created to raise specific mathematical notions: the latter are constructed within a teacher-pupils interactive process, aiming at objectifying mathematical concepts and facts by reflecting upon the exploratory and investigative processes that brought to them.

In this perspective, the teacher comes to play a multifaceted and complex role. He/she will need to play a wide range of roles (provoker, maieutic agent, orchestrator of discussions, model, etc.) in the classroom and thus will need to face a number of unpredicted and not easily manageable situations.

In particular, teachers will carefully: 1) plan teaching sequences able to foster students' conceptual constructions; 2) create an environment that favours pupils' mathematical exploration and formulation of conjectures; 3) choose suitable communicative strategies that support students' interaction and sharing of ideas. Moreover, with relation to specific mathematical issues, they need to be able to predict pupils' reactions to particular questions, their possible thinking processes and, most importantly, will need to face threads of mathematical discussions that deeply differ to what they expected. For this reason, teachers need to be able to foresee the problems that can rise in the classroom even before getting to solve them.

This stresses the importance of issues related to *teacher education and professional development*, which have been widely explored by researchers in recent years. This is also confirmed by the space devoted to this theme in the most important international congresses, by the increasing number of research studies on the teacher's role (Sfard 2005, Adler & Al. 2005) and, significantly, by the recent implementation of the 15th ICMI study '*The professional Education and Development of Teachers of Mathematics*' (Ball & Even 2008).

Researchers agree on the fact that in order to implement a socio-constructive type of teaching, not only should teachers get a fine and in-depth knowledge of the so-called 'pedagogical content knowledge' (Shulman 1986), i.e. notions related to learning difficulties, psycho-pedagogical issues and epistemological obstacles of the mathematical contents to be taught: most of all, they should get to know interactive and discursive teaching models (Wood 1999).

Regarding the latter issue, back in the ´90s some scholars highlighted the macro-effects on usual classroom activities caused by sudden and out-of-control micro-decisions made by the teacher (Artigue & Perrin-Glorian 1991).

In view of a reduction of these phenomena, several scholars stressed the importance of a critical reflection by teachers on their activity in the classroom: this might help them develop an attitude characterised by constant self-control in action as well as in the awareness of the possible consequences of their actions (Mason 1998; Jaworski 1998, 2002, Lerman 2001, Shoenfeld 1998). Mason, in particular, proposed the study of the *discipline of noticing* (Mason, 2002): he claims that the skill of consciously grasping things comes from constant practice, going beyond what happens in the classroom. He recommended the creation of suitable social practices in which teachers might talk-about and share their experience (Mason 1998).

In the last decade, several studies have been carried out both *with* and *for* teachers, in order to lead them to acknowledge the incidence of their choices (both actions and omissions) in the development of a mathematical discussion and, at the same time, make them aware of their way of being in the classroom. These studies show many different approaches, but generally they all aim at fostering teachers' critical review of their own conceptions of mathematics and of its teaching, so that they can become aware of the complexity of the classroom work, as well as acquire new and more appropriate models of behaviour (Borasi et al. 1999, Ponte 2004, Potari & Jaworski, 2003).

This trend of studies provides a framework to our investigation, although we also deal with issues concerning the renewal of the teaching of algebra through a linguistic and constructive approach in the sense of early algebra (for the latter theme, see Malara & Navarra 2003, Malara, 2008).

Our research experience with teachers made us aware of the difficulties they meet in both designing and implementing a socio-constructive type of teaching. We have observed how, despite the good intentions, in the development of mathematical discussions, often teachers do not devolve problems to pupils: in this way, students are not aware of the fact that the solution should emerge from a collective investigation, though validation and critical merging of each one's contributions. Teachers tend to talk to single pupils individually and do not co-ordinate peer-to-peer discussions: anxious to get to a conclusion, teachers often ratify the validity of productive interventions without getting the classroom to discuss and validate them. Moreover, teachers tend to let interesting contributions drop, if they diverge from the plan they have previously outlined, or rather are not able to recognize potentialities of certain pupils' interventions (Malara 2003, 2005, Malara & Al. 2004).

For these reasons, in accordance with Wood (1999), we deem important that both pre-service and in-service teachers, analyse their own or others' didactical processes, undertake new modalities for teaching mathematics and reflect and share ideas about their own actions.

OUR METHODOLOGY

Our studies focus on the design and experimentation of innovative teaching pathways and they have always been realised in a strict collaboration with teachers, following a traditional Italian model (Arzarello & Bartolini Bussi 1998, Malara 2002) which shows analogies to the co-learning partnerships model implemented by Jaworski (2003).

Our first studies mainly focused on pupils' learning difficulties, although we always kept an eye onto curricular innovations; later, the setting up of postgraduate schools for teacher training and the international trends in research led us to focus on teachers' problems and their education, with the aim to identify instruments and methodologies that may help them acquire the skills needed to implement a socio-constructive type of teaching, meeting the current cultural needs.

Our hypothesis is that necessary condition for teachers to become aware of the new role they should play in the classroom, of the processes enacted by a collective mathematical construction and the related variables, is a critical and reflexive observation and analysis of socio-constructive classroom processes.[1]. We also believe that this activity should be supported by the study of mathematics education theoretical results, which can strengthen teachers' knowledge of both the discipline and its

[1] Due to recent changes in the modalities of judicial proceedings, nowadays it is widely recognised in the legal field that the training of Public Prosecutors and Defence Lawyers should include the study of paradigmatic examples of trials. The aim is to explicitly show the effects of moves, decisions, actions undertaken during a trial and allow trainees to master a range of models and attitudes that mat suit their role in different situations, by examining different cases (see for instance Carofiglio, 2007)

teaching, support or modify their beliefs, and make them aware of the incidence of theoretical studies on their own professional development. (Malara & Zan, 2002).

For this reason, our current studies focus on the *analysis of classroom processes* that develop along planned sequences implemented by teachers. Our main aim is to lead the involved teachers to get a higher and finer control of their own behaviours and communicative styles and observe the incidence of critical analysis discussions on both classroom processes and pupils' behaviours and learning. Moreover we aim to design tools for teacher education that may be used in both postgraduate schools for training teachers and in distance education (Martellotta & Al. 2006, Malara & Navarra 2007, Malara 2008).

To reach these aims, we carry out a complex activity of critical analysis of classroom processes' transcriptions and reflect on them, looking at the relationships between knowledge constructed by students and teacher's behaviour in guiding them to achieve such constructions.

The activity develops along three subsequent phases, focusing on: the class teacher's autonomous reflection; teacher and researcher's joint reflection; shared reflection of teachers involved in the same teaching sequence; teachers and researcher(s)'s interactive reflection.

In the first phase, concerning individual interpretation of what happened in the classroom, the teacher is asked to transcribe mathematical discussions and comment upon critical or productive points. The teacher is thus forced to critically re-examine the process evolution with a particular focus on his/her own ways to communicate with pupils (asking questions, giving directions, making decisions etc).

Transcriptions of recordings, enriched with these initial comments made by the teacher, form the kernel of the *process diaries* (in the following simply 'diaries').

In the second phase, the researcher reads the diary analytically and writes both local and general comments. Diaries containing these new comments are then sent to the teacher by e-mail and later jointly studied by teacher and researcher. The latter guides the teacher to reflect on specific issues, by asking him/her to explain the meaning or reasons underlying some interventions, points to potential strategies to overcome dead-ends or rather explains some subtle facts of the emerged mathematical issues. The researcher also encourages global reflections on what has been done and highlights meaningful points in the evolution of the mathematical construction.

This joint analysis helps the teacher spot his/her habits, stereotypes, conceptions and to disclose possible gaps or misconceptions in his/her mathematical knowledge. This is a particularly important moment for the teacher, who becomes aware of the validity of his/her choices and teaching actions, by reflecting on wrong choices, omissions, misunderstandings, etc.

The third phase allows teachers to share what happens in their classrooms; it helps teachers to express their doubts or seek reasons underlying common teaching and learning problems. In this phase, teachers become aware of the diverging nature of the teaching processes they have carried out and reflect upon their own ways of interacting with pupils (interventions/silence, talking turns, reintroductions, timings).

In the fourth phase, teachers and researcher(s) collectively reflect on the points emerged from previous phases. Reviewing critical points of their own experience leads teachers to individually evaluate the global efficacy of their own actions, to make explicit obstacles, deviations, mistakes as well as new knowledge about their role. The researcher is thus enabled to observe how differently the experience influences each teacher and how the single personalities impact on the enacted educational process.

MULTI-COMMENTED DIARIES

Within recent projects[2] we made an interesting, although demanding and time-consuming, change to our methodology.

Due to the mentors'willingness as well as to participants' diverse locations, and to the need to share the analysis of currently studied processes, we decided that an ongoing analysis and discussion of the *diaries*, i.e. transcriptions commented upon by teachers, should be carried out by at least three people: *the mentor assigned to the teacher (M1); the co-ordinating mentor (M2); the head of the project (M3)*.

Diaries are thus enriched with a multiplicity of written comments (sometimes added independently on one another, sometimes following a hierarchical order), which reflect a wide and various range of points of view and interpretations, highlighting crucial points of the process as well as critical elements in the teacher's behaviour. Not rarely, starting from these comments, a teacher feels the need to get back to the diaries to mend his ways providing motivations or rather to clarify specific points, by explicitly pointing out hidden processes, behaviours of certain pupils etc. in order to support the reasons underlying his own decisions.

Multi-commented diaries become a complex investigative tool for both teachers and researchers. With particular relation to the examined processes, they can be characterised as:

- *formative tool* for teachers, enabling them to develop skills and sensitivity, and therefore improve the global quality of their own teaching action;
- *diagnostic tool* for researchers, enabling them to identify malfunctions in the teaching action, to formulate hypotheses and enact interventions to fix them; it also helps researchers to identify points for further research;
- *evaluation tool* for both teachers and researchers, providing elements to empower their interventions in the respective fields (teaching activity and design of local and/or general training and formative interventions).

More generally, they turn out to be a rich source for the production of materials to be used within laboratory-based training activities.

In order to have an idea of the materials generated by this collaborative work, the initial excerpt from a diary is reported in the Appendix: the diary refers to a teaching experiment carried out in a 6th grade class, with a young temporary teacher. The excerpt can be immediately put in context and easily read, thus giving an idea of how a diary is structured. It was chosen because there is a wide range of comments in it, that

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[2] The projects are Comenius *Professional Development of Teachers Researchers (PDTR*) and the National project '*Master in Science Education' (MDS).*

highlight problems recurring in all diaries, although with different nuances. Moreover the teacher, who did not make comments at the beginning, added some reflections on the basis of mentors' comments: the reflection process she went through is thus well documented.

Features of comments emerging from the diaries

Our investigation on the types of comments led us to identify five, interconnected, key areas. Some of them raise some points for further research and open new perspectives for teacher education:

1. General cultural and/or educational issues (e.g. conceptions of arithmetic and algebra, conception of teaching and pupils, conceptions about meaningfulness and centrality of certain topics).
2. Mathematical and educational-mathematical issues (e.g. sequences: what are they, how to teach them, how to represent them, what teaching critical points do they raise?).
3. Bifurcation between theory and practice (e.g. difficulties in realising what has been studied or planned, and in working on the basis of relational thinking).
4. Linguistic issues (massive use of operative linguistic expressions coming from the received model of teaching; difficult balance between colloquial language and language of scientific teaching; scarce attention to word paraphrases in view of an algebraic translation).
5. Management of classroom discussions (dialogues mainly between teacher and pupil; widespread prompting; yes/no questions; lack of attention to the development of 'social intelligence' in the classroom).

Two issues seem to be crucial and dramatic at the same time: *the teacher's language* in communication, often imprecise, not correct, full of slang expressions and rich in not always appropriate metaphors; *the conception of mathematics*, too often *operative*, as 'calculate' and 'find' often prevail over 'represent', and 'doing' over 'reasoning' and 'reflecting'.

Examples

In order to exemplify, we present here excerpts from multi-commented diaries, with relation to some of the comments' categories illustrated. Excerpts refer to teaching experiments carried out with grade VI classes within a teaching sequence centred on the study of numeric and figural sequences that can be modelled algebraically. The teaching sequence is designed by a group of teachers involved in the mentioned projects, after a theoretical study of papers from the international literature concerning issues related to the teaching and learning of algebra, with a particular focus on generalisation and algebraic modelling, and more generally to the teacher's role and the relationship between theory and practice.

The sequence develops along the exploration of five situations mainly concerning linear sequences, but it ends with a situation[3], in which pupils are supposed to carry out

[3] It is the 'apple trees' question from the International test P.I.S.A. 2000. It was selected by teachers within specific meetings concerning the study of the test.

a simultaneous exploration of two sequences- a linear and a quadratic one- together with a comparison of their progression. The teaching sequence's main objectives are to lead pupils to acquire a functional view of sequences and be able to construct algebraic representations, by modelling the relationship between ranking (or place) number and correspondent term of the sequence. Crucial mathematical points in the teaching sequence are the identification and representation of correspondence laws in general terms. These are connected with the enactment of different numeric representations of the sequence terms and their co-ordination, the recognition of structural analogies, the identification of variables and their naming through letters, the condensation of analogous arithmetic formulae in one single representation, the transformation of arithmetic formulae in order to recognise they are identities, the awareness of the role played by arithmetic properties in transforming formulae.

Example 1: General cultural and educational comments referring to an excerpt from a discussion, showing how the teacher paid little attention to pupils' interventions, viewed as trivial or not plausible

The teacher had asked pupils to explore the sequence with initial terms 4; 11; 18. The class had already identified the sequence's recursive generating law. The teacher writes the following table on the blackboard and opens up a discussion to introduce the class to the study of a representation for the general correspondence law.

Sequence ranking number	Sequence number	Operations made to jump from the place number	'Mathematical recipe' to construct the number
1	4	4	
2	11	4 +…	
3	18	4 + … + …	
4	25		
5	32		

Teacher: Operations made to jump from the first number. So what shall we do? Christian. You see we start from 4. In the second place, how do we get to 11? We make 4 +…
Christian: Eeeeeeeem….
Teacher: How do we get to 11? We make 4 + … ?
Christian: Three?
Teacher: To 11? Sabrine. **(1)**
Sabrine: + 7.
Teacher: We make 4 + 7. What about the third place, Sabrine, we make?
Sabrine: 4 + 7 + 7.
Justice: Wouldn't it be better to make 4 × 2? **(2)**
Teacher: What about the fourth place?
Sabrine: 4 + 7 + 7 + 7.
Teacher: What about the fifth?
Sabrine: 4 + 7 + 7 + 7 + 7.
Teacher: What if we had a sixth place?
Sabrine: 4 + 7 + 7 + 7 + 7 + 7.
Teacher: Correct. So, now we find…
Andrei: I didn't get it. What do I put in the first place?
Teacher: Well, there is 4 in the first place.

Andrei: I put 4 × 1. **(3)**
Teacher: Well, but there is no '×' there. The first place is 4 **(4)**

Mentors' comments (T stands for 'Teacher')

(1) M3. *This fragment of discussion highlights two points for reflection:* the issue of the whole class participation. *Cristian, with his doubts and his answer clearly shows he is 'elsewhere';* the teacher's behaviour*, since she 'moves forward' without paying attention to the pupil.*

(2) M1. *Why doesn't T comment upon Justice's intervention? M2. I wondered the same too. Perhaps Justice's intervention was lost in the mass of interventions. Transcriptions are important for this too, because they enable the teacher to reflect a posteriori.* M3 *I agree. Justice grasps a regularity but doesn't express it correctly, instead of saying 4 + 7 × 2 he packs everything in 4 × 2. T should have clarified this. She also missed the chance to introduce the multiplicative operator that allows for an objectification of the 'number of times' (that you need to add + 7 to the first term to get the considered number) as 'second factor' of the multiplication in the abbreviated representation of the additive expression which gives the number.*

(3) M2. *Also this intervention might have been investigated. What is Andrei's background thought? Why does he think about the product of 4 and 1?* M3. *Again we are in front of a badly expressed intuition. The student probably wants to 'fill the gap' he sees in the representation of the first term as compared to the others. Here T misses the chance to change the representation of the first term, 4, into one that fits with the situation, for example writing 4 as 4+0 and getting back to the class posing the problem to find a representation for the first term, similar to the other ones.*

(4) M3. *This intervention by T suggests that she excludes the possibility of representing 4 in another way, thus showing little algebraic farsightedness. It would be extremely appropriate to encourage these intuitions, although imprecise, trying to redirect them. In this case one might work towards a representation of 4 in terms of a general rule, for instance if the correspondence is modelled according to the law: "the term in place n (n-th term) of the sequence is given by 4 + 7 × (place number – 1)" this immediately leads to a representation of 4 as 4 + 7 ´ (1 – 1) i.e. 4 + 7 ´ 0.*

Teacher's reflection

All these remarks make me think I am really close-minded and I didn't realise before. I don't know whether this is a matter of attention, of being used to seeing things in different ways, of fear to get out of the scheme to be followed or the one I thought I should follow.

Example 2. Educational-methodological comment referring to a typical teacher' s action: the putting words into the pupils' mouth

The teacher deals with the transition from the recursive law to the general one.

Teacher: The first place is 4. Then, the second I make 4 + 7, in the third 4 + 7 + 7 and so forth. Let's see if I can use multiplication. How can I get to 11? I can make 4 + 7 but I can also make 4 +… **(1)**
Teacher: Biagio?
Biagio: 4 + (7 × 1).
Teacher: Correct. Because we saw that in order to get to 11… Biagio, you explain.
Biagio: Because making 7 × 1 is still 7. Therefore you make 7 × 1.
Teacher: Therefore we saw that to get to 11 we must make 4 + 7, but as you say, 7 equals…**(1)**
Biagio: 7 × 1.
Teacher: Hence saying 4 + 7 or saying 4 + (7 × 1) is the same. So what do we put in the third line Biagio?
Biagio: 4 + (14 × 1).
Teacher: Careful **(1)**

Biagio: $4 + (7 \times 1) + (7 \times 1)$.
Teacher: You do this. Any other idea?
Riccardo: $4 + (7 \times 2)$.
Comment

(1) M2. *I would advice T not to ask questions that 'invite pupils to complete the sentence', 'phoning' the answer. This strategy does not pay. It reminds me of an episode quoted by Brousseau (if I am right, taken from the comedy 'Topaze' by Marcel Pagnol) in which a preceptor is giving a French lesson to his pupil. Relatives assist quietly. The theme is: understanding from the context whether a certain word is in the singular or in the plural (in spoken French -s in the plural is not pronounced). The word is 'moutons' (muttons). The pupil has no idea and the preceptor is afraid his relatives might express a negative judgement. For this reason he walks around the pupil, who stays still, with the pen in his hand, whispering 'moutons'; then, not being successful, he starts to raise his voice 'moutons'... 'moutonss'... 'moutonsss'. Finally the pupil brightens and writes a –s at the end of the word. The surrounding gets calm . End of quote. M3. OK. Brousseau (1984) talks about 'Topaze Effect'.*

Example 3. Mathematical comments referring to both refinement and co-ordination of different laws that represent one single sequence

<u>Situation 1</u>

The teacher assigned the exploration of the arithmetic progression generated by the operator +7 starting from the term 4. The class had got to identify two 'rules', summarised by the teacher on the blackboard as follows:

4	11	18	25	32	39	46	53	60	67
(1) +7	+7	+7	…						
(2) 7×2–10	7×3–10	7×4–10	7×5–10	7×6–10	7×7–10	7×8–10	7×9–10	7×10–10	7×11–10

Law 1: you need to add 7
Law 2: I multiply by 7 then I take away 10

Comment

M3. The two laws are expressed very roughly. Attention, the first one is clearly a recursive law, the second one is general, although the variable is tacit. In order to compare them one needs to put them on the same level. The first law needs to be transformed into a general one. In order to do this, one needs to explicitly state the number of times that the operator +7 is applied, starting from the first term. The first law thus becomes

4 ; 4+7 ; 4+ 2×7 ; 4+ 3×7 ; 4+4×7 ……….. .

Pupils might also be led to notice that 4 = 4+0 = 4 + 0×7, embedding the first term into the general scheme. One might also try to proceed in parallel with the two representations:
law 1: 4 + 0×7 ; 4+7 ; 4+ 2×7 ; 4+ 3×7 ; 4+4×7; ……….. .
law 2: 7×2–10; 7×3–10; 7×4–10; 7×5–10; 7×6–10; ……….
inviting pupils to see the terms as correspondently equivalent and leading them to find out the underlying reason in the fact that, since 7×2 = 14 then 4 = 7×2-10.

<u>Situation 2</u>

The teacher compares the representations of the sequence's terms following the two identified laws (situation 1). From the analysis of cases in the classroom, they get to the following conclusion:

Posto	Numero	Regola 1	Regola 2
1°	4		$7 \times \boxed{2} - 10$
2°	11	$4 + 7 \times \boxed{1}$	$7 \times \boxed{3} - 10$
3°	18	$4 + 7 \times \boxed{2}$	$7 \times \boxed{4} - 10$
4°	25	$4 + 7 \times \boxed{3}$	$7 \times \boxed{5} - 10$
8°	53	$4 + 7 \times \boxed{7}$	
12°	81	$4 + 7 \times 11$	

In rule 1 the changing number is teh third, i.e. the second factor. This number changes after the place , i.e. it equals the one in the previous place.

In rule 2 the second factor is changing again, but this time the place is equal to the number precedine the second factor.

Comment

M3. The rules are not very clear from a linguistic viewpoint. A better statement would have been 'according to rule 1, when a term of the sequence is represented, the third number changes, i.e. the second factor of the written product. This number changes <u>*in correspondence*</u> *with the place number and it equals the place number minus 1 . One remark: why wasn't the first case completed? (A pupil had seen it, it is not appropriate to throw away fine and valuable interventions like this one). Another remark: I regret to notice that rule 1, although identified, remains not expressed and not objectified. Rule 1, generated by a re-writing of the sequence terms through successive multiples of 7, needed to be made explicit, at least verbally, and pupils should have taken charge of this task. We would have probably got formulations like 'the sequence number at a certain place is 4 plus 7 for the first place number minus 1' (faithful translation of the procedure) or rather ''the sequence number at a certain place is given by the product four plus the place number minus 1 times 7" (mixed relational-procedural formulation) and others; the teacher might get to express it in the evolved relational form: ''the sequence number at a certain place is the sum of 4 and the product of 7 times the place number minus 1). I purposefully avoided using terms like preceding or antecedent of the place number, of little help in the algebraic translation. This variety of verbal formulations already leads to interesting problems to be discussed in the transition to the algebraic formulation (in this, Brioshi's metaphor works perfectly): these problems not only concern the representation of the variable (place number) but also the use of parentheses.*

I'd like to remark that rule 1 is given by expressing the second factor as a function of the place number (working well for the algebraic translation). Rule 2 was expressed by giving the place number as a function of the second factor (not working for the algebraic translation).

Example 4 Comments related to linguistic and representation issues in the transition to generalisation, within a discussion which was heavily influenced by the teacher's language

The class had identified the correspondence law between natural numbers generated by the operator "+7" starting from 4. They then went on to tackle the generalisation problem and worked collectively on the meaning of the term 'n-th'. The discussion reported below centres around the search for a formula that might represent that correspondence. The teacher tackles the question with pupils, by summarising on a table all the results they had got to. In the study of the 30^{th} place case, there is a mistake: within the generalisation process, the number preceding 30 is swapped with the number following it.

Posto	Numero	Operazioni…	Regola 1	Regola 2
1°	4			
2°	11	4 + 7	4 + (7 × 1)	7 × 3 – 10
3°	18	4 + 7 + 7	4 + (7 × 2)	7 × 4 – 10
4°	25	4 + 7 + 7 + 7	4 + (7 × 3)	7 × 5 – 10
30°			4 + (7 × 31)	
n				

Teacher: *I want to know: if I'm at place n, that we said – remember? – it was a place at a certain point, without knowing what point it was. Well, I want to know what is the rule that allows me to find this number at position n* **(1)** [I point to the n-th term on the blackboard] *right*? *Good, so, let's find the rule* **(2)**. *Benedetta*?

Benedetta: *Ehm, because I thought that n-th was the last, then I wrote "there isn't any, because the sequence is infinite* **(3)**".

Teacher: , *Alright, this is a true remark, and it might be helpful later, we keep it apart* **(4)** *So, how can we find the formula we will need?* **(5)**! *Don't look at me, look at your sheet and the blackboard! How can you find it?* **(6)** *Andrea?*

Andrea: *Well, if we know… last time we said n-th stands for any place* **(7)**

Teacher: *Question: number n means a number at any place* **(8)** *without telling you what number it is, this is the difficult part! What formula do I write for the number at the n-th place*? **(9)**

Sergio: Well, *I think you can't find it because n-th is a number you don't know.*

Andrea: *As you said, n-th stands for a number at any place, so I say like Sergio, if the place is not definite, we will never know what number it is!* **(10)**

Teacher: *Correct, I agree too! If I don't tell you, at the 3rd , at the 4th, at the 100th, at the 7003rd place, we won't be able to know. But if I tell you that this number ... about this number, instead of telling you the place number, I tell you it is at place n, can I make a calculation ... can I write a formula to find this number?* **(11)**

Stefano: *I think so... we don't know what number is n and even though we don't know it we can find a formula.*

Sergio: *I think we can't find it because there is no precise formula* **(12)**

Teacher: *Mmh, but in your opinion, this formula exists and we can't understand it, or rather it doesn't exist?* **(13)**

Alessandro: *In my opinion… I don't know…*

Andrea: *As to me... I found a formula for the n-th place, but I don't know whether it is correct … To find the number at the n-th place we should make 4 × 7 × n.*

Teacher: *Are you sure? Look at the other formulae… you missed out...*

Francesco: *He said 4 × 7 and it should be 4 + 7*

Teacher: *Ok, it was an oversight!* **(14)**

Tamara: *I tell Andrea: how can you find 4 + 7 × n-th place if you don't even know what it is !*

Teacher: *This is exactly the question we asked ourselves up to now. But I don't want to know what number is there, I don't want to know the result of the calculation, even at high school you won't be able to do 7 × n if you don't know what number is 'n' ! But I only wanted to know the formula and Andrea proposed one! It's now clear that we can't find this number! You were right to underline it. Now I want to know, is the formula correct?* **(15)**

Tamara: *I don't think so*

Benedetta: *I think we are beating about the bush. Sorry... but we are doing something which doesn't exist!!! Why are you asking so many questions!! Stop!! There isn't any!!*

Carmen: *Benedetta is right!!!* **(16)**

Tamara: *I don't think so, because if you look at the other examples... you see that 4 + 7 × 4.. I mean instead of doing the n-th place … there is 4*

Teacher: *But at the 4th place there is 4 + 7 × 3, therefore the second factor is …?*

Tamara: *3.*

Andrea: *if we look at the number at the third place, the operation written there is 4 + 7 × 2* **(17)** *so we can deduce that the number that changes is the same as the place, but minus 1 1.*

Teacher: *Tamara, Andrea was saying "place minus 1"… how do you call that number?* **(18)** *the…?*

Tamara: *Eh?…*

Teacher: *We also wrote it on our poster* [one and the operations*: a – 1* was written as the number preceding a, with $a \in N$ and $a \neq 0$] **(19)**

Tamara: *Ah… the predecessor*

Teacher: *The predecessor, as the king that comes before*

Sergio: *The preceding.*

Comments

(1) T. I now realise I have used wrong terms thus inducing students to give the answers they have given and I desperately "fought". When I said "the rule to find this number at position n" students understood I wanted to know the value of a_n. Perhaps I should have said "the rule to find one sequence number, knowing its position ".

(2) M2. My suggestion is to lead the class to discover, and highlight with arrows, relationships, repeated numbers, 'local' regularities. Many of these might not be productive, but help pupils get used to carry out full explorations. For instance, the same sequence, proposed in another class, led some pupils to identify a relationship between numbers in the first two columns and represent it with $11 = 2 \times 7 - 3$, $18 = 3 \times 7 - 3$, $25 = 4 \times 7 - 3$, and so forth. Arrows might link the various fours with the first term of the sequence, numbers 1, 2, 3 of the fourth column with the place numbers of the first column, shifted one line down, etc. These latter arrows might highlight the fact that 31 is wrong and it should be substituted for 29.

(3) M1 Attention: the factor (n-1) corresponds to the n-th place, which, according to Benedetta, gives the second-last number.

M3. Benedetta contradicts herself, if the sequence is infinite, places are also infinite and n can't be the last. Perhaps she means n as a 'very great' place-number. With this contradiction she expresses her belief that a number at a non-defined place cannot be represented. But she has not the meaning of n as indicator of a number that we don't want determine.

(4) M3 It's true that the sequence is indefinite but it is NOT true that 'n' means 'last'. We can't state the truth of a proposition made of two propositions, one of which is false. Attention to logical aspects of language!

(5) T. Due to the silence, I chase the audience.

(6) M1**.** T. is tormented by the idea she should get to the formula written in algebraic language. I keep thinking that in this phase the objective is to lead pupils to grasp the relationship between place and correspondent number, and clearly express this relationship.

M2 Why not making them express things in natural language, by describing the shape of columns 3 and 4: "I get the number by adding to the initial number as many 7 as ..." or in any other way. The paraphrases proposed by pupils can then be compared and the most suitable to be translated into algebraic language for Brioshi, be chosen.

(7) M1 Very well Andrea, that "any place" is gold!

(8) M3 More than 'any', term which triggers the idea of variability, it would have been more appropriate to point out that this is a number we don't want to specify, 'indeterminate' (and this term, while focusing on the element, somehow fixes it)

(9) M1 I often wondered why we shouldn't put, in the mathematical recipe's column, the – mental or not- operations done to identify the factor that multiplies 7, starting from the number at the given place. In this way, pupils would have grasped the regularity, the re-iteration of a procedure, getting closer to the construction of the formula. M3. the formula can be determined by identification in the studied cases of invariant parts (4+ 7×…) and variable parts (place number – 1).

(10) M2 Approaching the use of letters is very complex, it requires lengthy times, different strategies, comparisons and explorations and involves continuous and unpredicted evaporations. Intuitions of different meanings co-existing in the interventions made by Sergio and Andrea are inevitable and physiological. The (real or presumed) need to conclude and get to the rule might have imposed the teacher a speed that can hardly go together with this complexity. We are fully immersed in algebraic babbling, and learning a new language, with its meanings and rules necessarily requires settling (metaphor: sedimentation of solid substances dispersed in liquids).

(11) T. I understand now why they could not answer! We don't understand each other! As I said before, the verb "to find" puts them on the wrong track! Perhaps I should have said "find a representation of the place number n that makes us understand that this number is in the sequence". Too complicated! I don't know…

M1 I agree on the damages caused by the term "to find".

M2 I agree on representing too, even more if also this term (Glossary) becomes one of the keywords of the class' cultural baggage and therefore it gets a negotiated and shared meaning (again Glossary).

M3 Finally, Good T! Representing, yes, representing is the key term.

(12) M1 Sergio's words are to be questioned. M3. I agree

(13) T. It is an important moment, Alessandro cannot explain, perhaps an example would have been appropriate.

(14) M3 Here T should have taken the chance to verify Andrea's formula for already considered values of n: 2, 3, 4. 30. In this way the students would have been able to grasp the sense of the formula, mend the case 30 and 'get out of it'.

(15) M1 Requiring the formula makes students panic, it's clear. I insist, I would have asked "How would you find the number at place n? Then I would have written, from dictation, pupils' indications: I'm sure they would have dictated the notorious formula.

(16) M2. To the Bastille! To the Bastille!

(17) M2. I repeat already expressed concepts: I get more and more convinced that it is necessary to lead pupils toward the skill of seeing a written expression like 4+ 7 × 2 from an algebraic point of view, going through the overcome of the arithmetic viewpoint of 'making operations'. It would be a great result from an educational viewpoint to be able to guide pupils to conceive a non-canonical form, like (4+7×2) as a representation of a number (18), and therefore as a representation of relationships (additive and multiplicative) between numbers, independently on the actual calculations. Thinking of the written expression above as 'essentially' a sequence of operations, provides the expression itself with a sort of 'temporary character' which blocks any interpretation as mathematical object, at metacognitive level.

(18) T. I found it difficult along the whole activity. I now realise that my questions were not always focused: in this case, I should have stressed that it was about a <u>relationship</u> between place number and rule 1, and I might have asked Tamara "Andrea identified a relationship between rule 1 and place number n and talked about a certain – 1, how may we express this in mathematics?". However, due to this kind of difficulty I gave up the idea of proposing the ArAl unit about Brioshi to this class.

M2. I think that this difficulty (perhaps it is a difficulty you meet) should have opened the way to Brioshi.

M3. I agree,"representing" should absolutely substitute for "calculating".

(19) M3. I wonder how pupils interpreted this expression, from their behaviour up to this point, the letter was neither seen as variable, nor as indeterminate. Was it a label without a meaning?

SOME FINAL REMARKS

Commented transcriptions of classroom-based processes support the teacher's a posteriori reflections on how the activity was carried out and managed and require him/her to reconstruct it critically, through an interpretative effort that is highly formative. These diaries enable researchers and mentors to test teachers' consistency with their teaching practice, declared beliefs and reference to the theory at stake (both from mathematics and from mathematics education). Besides, they require an explicit *fine* analysis of micro-situations that show the teacher both consistencies and inconsistencies of his/her teaching action. Moreover, a global analysis of comments leads the class teacher to elaborate on the activity with significant follow-ups on his/her teaching practice, and enables mentors and researchers to detect in a deep and extensive way teachers' cultural backgrounds and attitudes.

The sharing of commented diaries and the analysis of the diverse whole-class discussions arisen from one single problem situation, leads to an objectification of the reasons that determined them. Moreover, by comparing their own actual implementation of a certain step of the teaching sequence to other colleagues', each teacher identifies important distinctive elements and reflects on both efficacy and limitations of his/her own work (for instance, hurried and decisive interventions, little attention to listening, lack of understanding of potentially fruitful interventions, scarce ability to orchestrate voices, difficulty to contain leaders or minimise the effects of tacit alliances, etc.).

All this leads teachers to acquire an increasingly deeper awareness of their own way of being in the classroom, a better control on their behaviours, and stimulates them to think about, and enact in a fine way, changes to their teaching modalities.

It is now appropriate to make some remarks on the validity of our methodology as well as on what emerges from the described processes. Due to the line-by-line comments made by two or three researchers to teachers' diaries, the transcription is analysed under a multiplicity of viewpoints, ranging from content-related aspects (setting up of the problem exploration, mathematical aspects developed, attained or missed objectives, ...), to communication and language-related aspects (Questions' formulation, operative guidelines given, ...), to issues related to the control of pupils' participation (number and type of interventions) as well as the agreed didactical contract (induced attitudes in pupils and socio-mathematical norms in the classroom). The analysis we carried out leads to 'freeze' a wide range of remarks and in-depth comments that reveal and amplify different aspects of a teacher's professionalism (his/her mathematical and pedagogical knowledge, his/her conceptions about teaching and relating to pupils, the style of the agreed contract, his/her hurries and digressions, his/her affectivity). It is a real 'radiography' of the teacher, in front of which (s)he experiences a moment of healthy crisis, very often followed by a positive reaction leading him/her to challenge him/herself and act towards a fruitful professional development.

Besides this, multi-commented diaries offer extensive material to construct prototypes of activities for critical reflection, useful in both laboratories and training activities concerning the specific taught discipline, useful to novice teachers, in distance teacher education, to the construction of learning objects, in the formative training of mentors and supervisors, to provide them with models of analysis of interactive and discursive educational processes.

These tools were used with interesting results in SSIS (pre-service teacher training courses, in-service teacher training courses, e-tutoring, ...). Observation of teachers tackling tasks of this type clearly shows the main objective, which is to put the teacher in situation, so that (s)he might link critically three crucial points: own conception of the mathematical topic at play (and of mathematics itself): the conflict caused by the meeting-clash with the teaching modalities used by colleagues and by the results they attained; a mediation between these two crucial points produced by a collective sharing and a dialogic relationship with researchers.

Analysing the comments made by researchers, the authors' epistemology appears clearly, as some particular types of comments prevail. Both agreed and diverse points of view in the produced comments help teachers, the former by strengthening comments themselves and the latter as their complementary nature provides an enrichment.

This methodology is obviously strictly depending on teachers' involvement and commitment. For this reason, it cannot be used during short refresher courses with significant results. However we believe that it is possible to spread curricular and methodological innovations in schools and society, through a serious and convincing dialogue between participants in long term projects (as our ones), their colleagues and also parents of their students.

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APPENDIX

INITIAL FRAGMENT OF A MULTI-COMMENTED DIARY

We report the initial fragment of a diary referring to the process of implementing the teaching sequence in a 6th grade class, carried out by a young temporary teacher. Due to the variety of comments it contains, this excerpt provides evidence of the complexity of a diary. In particular comments refer to:

a) general aspects of issues related to teachers' beliefs about the usual type of mathematical activities, and the degree of relevance of the activity proposed to pupils, in terms of the 'minimum curriculum' expected by both school and parents;
b) representation-related aspects, that involve mathematical issues about potential infinity and generalisation;
c) didactical-methodological issues regarding didactical contract, relationship with pupils and testing of pupils' actual understanding and learning,
d) linguistic issues linked to both the teacher's communicative style and the attitudes induced in pupils.

The teacher proposes a topic concerning the exploration of a sequence, given the first three terms (it is the arithmetic progression with initial element 4 and step 7). The activity is aimed to determine a general representation of the sequence. The excerpt reported here deals with pupils' appropriation of the task. Analytical comments, which provide the structure of the diary, are reported in the footnotes in order to preserve the discussion flow. Authors of comments are labelled as: T: teacher; M1: reference mentor; M2: co-ordinator of mentors; M3: head of research.

Teacher: *Today we are going to do something different*[4]
Some*: Great!*
Riccardo: *What are we going to do?*
Teacher: *We play a game. Has any of you done the games you can find on puzzle magazines?*
Chorus*: Yes!/No!*
Teacher: *It is a magazine where you find crosswords and other games and puzzles.*
Teacher: *I write some numbers and then I put some empty spaces, which means there will be other numbers:* 4 11 18 __ __ __*. Ok*[5]*?*

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[4] M2. The 'diversity' of this activity is presented as a motivating aspect. This is partially true, but it can also represent a sort of distracting factor ('Great!' in the subsequent intervention induces this suspect). On principle, I consider neutral approaches, that actually introduce pupils to a 'permanent sharing' and do not emphasise the episodic nature of the activity, more productive.
M3. I agree, even though I see this 'opening' as a support to the teacher, who is aware she is undertaking a different activity, somehow risky and far from usual ones.
T. It was actually meant to be a way to capture everybody's attention, including those who usually don't follow or find it difficult to stay focused. But it might have been a sort of unintentional psychological crutch.

[5] M1. I would have clarified some conventional items: empty spaces (I imagine dashes) that will be filled by numbers, dots to indicate that numbers might be a lot, infinite. This would have been useful to distinguish between the concrete signs on the blackboard (with its space constraints) and the abstract nature of mental images.

Try and think about which should the numbers be.[6] *We cannot put some random numbers, we should try to explain why we put those numbers.*

Lorenzo: *perhaps I got it, … I know it, I know it*

Teacher: *Lorenzo, since you said "I know it, I know it, I know it", say it aloud.*

Lorenzo: *4 + 7, 11; 11 + 7, 18; 18 + 7, 25.*

Teacher: *So which numbers do you add?*[7]

Lorenzo: *18 + 7, 25; 25 + 7, 32; 32 + 7, 39; 39 + 7, 46. (At the blackboard, the teacher writes in columns the following equalities: 4 + 7 = 11; 11+7 = 18; 18 + 7 = 25; 25 + 7 = 32; 32 + 7 = 39; 39 + 7 = 461 + 7 = 18)*

Teacher: *Etcetera. Does anyone disagree?*[8]

Antonio: *How did he do that?*

Sabrine: *I didn't understand.*

Teacher*: Didn't you? So, Lorenzo explain to Sabrine what you did*[9].

Lorenzo: *I summed 4… it's a chain. 4 + 7 is 11. 11 + 7 is 18. 18 + 7 is 25. 25 + 7 is 32. 32 + 7 is 39. 39 + 7 is 46. I kept adding 7 to what I got*[10].

Sabrine: *Why do we need to add that 7?*[11]

Teacher: *Why do we need to add that 7, she's asking? Who's going to answer?*

Andrei: *Me, me!*

Teacher: *Andrei*.

Andrei: *Because in this sequence a number goes ahead by 7. Because it might go ahead by 10 in another one.*

Teacher: *Did you understand?*[12]

M3. I agree. I would have put at least 4 terms in the sequence, commas between place marks, dots at the end to indicate the sequence's indefinite length. These tricks might avoid limited visions, misunderstandings and overcome a view of finite sequence, inducing the idea of infinite sequence.

T. True. I actually took for granted that the written expression at the blackboard was clear to anyone: dots instead of missing numbers. I didn't think about pointing out the difference between the infinity of actually missing numbers and the finiteness of the blackboard.

6 M1 I would have solicited the class to identify what could connect the three numbers, what they share, that is: why were those three numbers chosen?

M2. I agree. Putting the activity in context might favour clarity in the didactical contract.

T. Right, I didn't even think about it. For sure, due to my scarce preparation to the activity, I did not reflect upon these possible aspects and went straight to the core of the problem I proposed.

7 M1. Better than this, which numbers might be included and why?

M2 Ok. It is important to make explicit that statements have to be justified .

8 M3. Is it taken for granted that the rule uttered by Lorenzo is clear to everyone? Is it rather an interlocution? The teacher might have better asked the class explicitly 'what rule did Lorenzo follow?' T. True. My question was meant to provoke their reaction. I would have later asked them to explain and justify both affirmative and negative answers, as it actually happened.

9 M1. I would have asked Lorenzo to explain what led him to construct the numbers following the given three numbers in that way ...

10 M3. Lorenzo expresses what he did starting from 4 (first term). In making a summary, he forgets to mention the starting point. T. should have intervened to invite him to be more precise. Example 'What do you mean by 'what I got', and 'where do you start adding 7?' T. True. I think I have preferred to let them carry on with their interaction in that moment. Pupils speak fast, overlap their comments without letting classmates finish their sentences. It's sometimes difficult to intervene and still let them follow the thread of the arguments they are trying to express with effort.

11 M2. Sabrine brings up a sore point and opens a door in the direction anticipated by M1 in previous comment. Great.

Sabrine: *No.*
Teacher: *Laura, try to explain*.
Laura: *Sabrine, the teacher gave some numbers: 4, 11 and 18. Try to count how many numbers you have from 4 to 11*.[13]
Sabrine: 18.
Laura: *So: 5, 6, 7, 8, 9, 10, 11. They are 7 numbers. From 11 to 18 how many do you have? There are seven numbers more, I tell you. And the teacher put some dots and you must discover the number that going ahead you were adding 7 more.*
Sabrine: *Ah! I got it*.
Teacher: *Did you understand?*
Sabrine: *Yes*[14].
Teacher: *Ok, so what is the rule?*[15]
Riccardo: *That you must find out how many numbers you need to get there*.[16]
Teacher:: *Well, give me a nicer rule*.[17]
Riccardo: *Now, you count how many numbers there are from one to the other and you go on like that.*
Giuseppe: *Depending on the given numbers, we must find the number which we go on to...*
Lorenzo: *Depending on the given numbers, we must find the instruction.*
Giuseppe: *And what is the instruction?*
Teacher: *In actual fact, this is the thing. When we must find a rule or explain something to someone, we need to make ourselves understood. Maybe everything is clear in our head, but what we say is not always that clear to others as well*.[18]

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[12] M3 This intervention is not very appropriate. What the pupil says is not clear and T should have invited him to be more precise.

[13] M2. Laura does her best to lead Sabrine to understand that there is a '7 units step' between one number of the sequence and another one.

[14] M1. Perhaps it would have been good to test, with an example, if Sabrine really understood.
M2. I agree.
M3. The dialogue sounds sterile. You can't accept a 'yes'. You need to get sure that the pupil understood and give her the chance to make the intuited procedure more solid. It would have been enough to ask the pupil what she would have put after number 18.
T. True, I should have verified. I usually do it, when I see some pupils having doubts, I assign them more exercises or ask them to give another example. I think that, in that moment, but also along the whole experience, I made the mistake to be in a hurry to finish everything I had planned. I cut off that issue, because I saw it as trivial (but I should have thought about them) and I felt I had already wasted too much time..

[15] M1. I would have posed the question as: may we talk about 'rule of behaviour'?
M2 T. should try to lead the class to a representation of the relationship between numbers in the sequence.
M3. I agree. The teacher could take the chance to intervene, focusing on a term and its successive and working on verbal expressions of the relationship that links them. The term 'rule' is jargon, relationship is a better one. T. I used "rule" because I thought this term was more straightforward and clear to everyone.

[16] M3. There, where? Is the task uniquely about gaps to be filled, for this pupil?

[17] M3. Jargon expression, the term 'nice' is used as a synonymous of 'clear'.

[18] M3. This intervention by T. is relevant and makes the contract clear. T. Thanks. This is to me the hardest problem we have to tackle when we want to get pupils to interact.