BEHAVIOUR OF 4th-GRADE STUDENTS IN FRONT OF A PROBLEMATIC SITUATION AIMED AT APPROACHING THE CONCEPTS OF FUNCTION AND INVERSE FUNCTION

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We face the question of an early approach to the functions, seen in the wider frame of the binary relationships, promoting the use of different registers of representation and their coordination. We trace the guidelines of one of our experimented didactical paths on this topic and we present the analysis of some excerpts of a classroom discussion where a problematic situation is modelled through the arrow, graphical and algebraic representations operating simultaneously with the direct and inverse functions. Finally we discuss about the roots of some difficulties met by the pupils and about some teacher's behaviour.

INTRODUCTION

The teaching and learning of the concept of function pose many problems which are due to the complexity of its history. As witnessed by many studies (Harel & Dubinsky 1992, Mavarech & Kramarsky 1997, Slavit 1997), this is reflected in the partial or distorted visions assumed by the students and in the difficulties and obstacles they meet. It must also be underlined that owing to the lack of a teaching tradition for this theme, even teachers present uncertainties and difficulties with this topic (Even 1993). Italian middle-school syllabuses privilege the study of relationships and the use of the algebraic language for their encoding. They focus the attention on functions as tools for the modelization of simple physical phenomena. This implies the prevalence of a vision of function as a law between quantities that can be represented algebraically, rather than the modern vision of arbitrary functional correspondence, which is delayed to upper secondary school.

We agree with Duval (2000) that in order to construct real knowledge it is necessary for the subject to grasp the difference between the mathematical concepts and their representations. We believe that the pupils should be guided during the learning process so that they are able to operate this distinction in a not a-posteriori way. So we consider important to approach to functions through the analysis of their different representations (words, arrows, tables, formulas, graphs, …) which contribute to build such concept in order to promote in the pupils a flexible and articulated vision of it. At the same time, we consider it important that students can get to conceiving a function as a particular set of couples of elements but also they can manage various representations of it being aware of their different features (for instance the fact that the arrow representation highlights in a dynamic way the covariance between the components, the graph gives a global static representation of it, the algebraic representation is comprehensive of both these aspects).

Usually, in our junior secondary school the link between the concept of relationship and the one of function is often neglected, neither a joint vision of the couple <function, inverse function> promoted, in coherence with that is done in the frame of the relationships, nor the connection among the involved representations is stressed (from the relationships to the functions one goes silently from the set-arrow representation to the graph-Cartesian one). Generally the didactical interventions focus on the linear functions, but domain and co-domain are not highlighted, and phenomena involving natural numbers as variables are very often represented with straight lines (with an implicit jump to the continuous numerical ambit). Moreover, usually it is not given the necessary care to the fact that the representation of a situation depends on the subject that one takes into account.

OUR HYPOTESIS

Our hypothesis is that at junior secondary school level this question has to be faced in a pervasive and gradual way, so that the pupils can:

- conceive the functions in the wider frame of the binary relationships, seeing them arising from realistic situations;
- study problematic situations involving more than two variables identifying in the same contest binary relationships related to different couples of variables, and activating various registers of representation (words, arrows, tables, formulas, graphs, sets of couples);
- give prominence to the coordination of different representations of a same relationship dwelling upon the interpretative aspects of the representations and, in particular, highlighting in the algebraic sentences or in the Cartesian graphs the representative elements of subject, predicate, object of the verbal sentences that they translate (it is generally spread among the students that the subject of the functional relationship is the independent variable).

Moreover, we consider it important that the teacher brings the pupils to operate with the couple <function, inverse function> from the very beginning, so as to overcome the stereotype of the inverse function as a subordinate of the given one, realizing their joint graphical representation on the Cartesian plane and highlighting the geometrical features of the two graphs.

Here we present a fragment of the didactical path of constructive kind (see below) which belongs to the current of our studies devoted to the approach to functions (Malara & Iaderosa 2001, Fiorini & Al 2006). These studies have been realized in the wider frame of our *ArAl project,* aimed at an early approach to algebra as language for modelling, solving problems and proving (Malara 2003, Malara & Navarra 2003, www.aralweb.it). In particular we shall focus on a problematic situation in relation to which we shall analyse excerpts of classroom discussions that highlight difficulties they met in the collective construction of the symbolic encoding of a functional relationship and in the coordination between graphical and algebraic representations of the couple <function, inverse function>.

METHODOLOLOGY

The work has been carried out together with a group of four middle school teacher belonging to the net of schools involved in the ArAl project, these teachers are not simply experimenters but are teachers-researchers in training according to our classification (Malara 2004) which characterizes the different roles of the teachers in the Italian model of research for innovation in mathematics.

The experimentation has been realized along the whole school year in four classes of 4th grades and has been preceded by the teachers' study of the theoretical frame of the ArAl project and of some papers concerning not only didactical questions related to the approach to functions but also papers on the orchestration of classroom discussion such as Cobb & Al. (1992), on the construction of mathematical knowledge in interaction (Hershkowitz & Al. 2001) and on the role of the teacher (Malara & Zan 2002).

The discussions carried out in the classrooms are recorded and transcribed by each classroom teacher, then they are critically analysed by these with the contribution of the researcher. The work of joint comparison of the transcripts allows to collect interesting protocols of classroom discussions which highlight not only different ways to face the questions by the students but also the incidence of the teacher' choices and beliefs on the students' behaviour.

SOME HINTS ON THE PATH

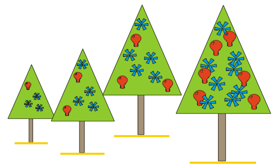
The path develops through a set of problematic situations involving several variables and it is posed through a text which is articulated in various questions and supported by iconic representations. Its first phase concerns situations involving qualitative variables and focuses on the verbal formulation of the observed relationships taking into account different subjects; the representations used are mainly arrows and sets of couples. The second phase develops around some realistic situations involving quantitative variables and it is devoted to the passage from the verbal representation of the relationships to the algebraic one, and more in general to the coordination among representations in various registers; several representations are used: arrow, tables, formulas and graphs. Within this phase of the path, starting from a comparison of different verbal formulations of a given 1 to 1 relationship, aspects of formal translation are highlighted, pointing out elements that represent subject, predicate and object in the algebraic formulae. In the Cartesian representation a particular care is given to the interpretative aspects, highlighting correspondences between subject (object) of the verbal representation and its location on the axes (students generally share the idea that the subject of a functional relationship is the independent variable). Moreover, in the case of invertible functions, the pupils are brought to tackle the simultaneous graphical representation of the couple direct and inverse functions. The third phase is focused on the interpretation of couples of algebraic sentences in terms of functional relationships as to a given situation. The last phase faces the question of the composition of relationships from an algebraic point of view and the link between functions and equations.
Here we report some excerpts of a classroom discussion which comes into play at a crucial moment of the path: the transition from qualitative to quantitative relationships, as it concerns the first approach of the pupils to the algebraic modelling of relationships and the coordination with their graphical representation in the Cartesian plane.

EXCERPTS OF A CLASSROOM DISCUSSION

The classroom discussion develops around the problematic situation reported below. It has been carried out in a 6^{th} grade class, in the middle of the school year (other problematic situations related to this phase of the path and faced by the pupils are reported in appendix.)
Up to that moment the pupils had explored different and sometimes complex open situations, involving several qualitative variables, getting to the identification of different binary relationships represented mainly by words or by sets and arrows.
We limit ourselves to reporting some excerpts of the discussion concerning the questions of constructing the algebraic representations of the direct and inverse functional relationships in play. The excerpts chosen are related to the difficulties met by the pupils on translating the representations from arrows into algebraic equalities and in coordinating the algebraic representations of the direct and inverse functions with their graphs in the same Cartesian plane. In the report, the classroom dialogues are intertwined by teacher's comments or reflections, written in Arial.

The problematic situation

It's Christmas. In a square in the city centre many colourful firs have been exhibited. Here are four of them chosen randomly
Which relationships do they hide?
Represent them with the arrow language.
Then invert arrows.
A big fir has 75 lamps. How many ribbons?
Another bigger fir has 220 ribbons. How many lamps?



The discussion

In front of the questions asked, pupils appear initially disoriented.

Giulia: I noticed that in the 1st tree there are 1 lamp and 3 ribbons, in the 2nd: 2 lamps and 4 ribbons, in the 3rd: 3 lamps and 5 ribbons and in the 4th: 6 lamps and 8 ribbons. There are always two ribbons more than lamps. This can help us answer the question.

The class agree. Conclusion: in each tree there are 2 ribbons more than lamps. *(The posed questions oriented pupils in the search for the relationship).*

Valeria: (*doubtful, she restarts the analysis*) in the first tree there are 3 ribbons and 1 lamp *(she is inverting the relationship without realising it and she does not agree with the previous conclusion).*

Francesco: 2+1=3, all right

Teacher: what are we putting in relationship**?**

Giulia: lamps with ribbons

Teacher: or better?

Giulia: the number of lamps with the number of ribbons.

Alessandro goes to the blackboard, represents the relationship arrow form and he tries to explain the meaning of arrows:

Alessandro : since in the first tree there are 1 lamp and 3 ribbons I connect 1 to 3.

Teacher: What does the arrow mean?

Alessandro and others: how many ribbons it has, how many ribbons more it has.

N*obody thinks about assigning the arrow an operative meaning. The jump from the analysis of qualitative relationships to numerical relationships is probably too sudden. They find it difficult to express y as a function of x, the subject changes.*

Teacher: What is the difference between the relationship we are examining and all the previous ones?

Many**:** it is a relationship between numbers.

Alberto goes to the blackboard, proposes the Cartesian representation, he draws the two perpendicular axes (only one quadrant). He labels axes with x and y.

Teacher*:* What do x and y represent?

Alberto: x represents the x axis,

Teacher*:* on the plane we have to represent the couples of quantities of the two objects on the various firs, which do you want to represent on the x axis?

Alberto writes: x = number of lamps

Teacher*:* Andrea what about y?

Andrea R.: the number of ribbons.

We write down the key. x = n. lamps; y= n. ribbons

… (for lack of room we skip an interesting part of the discussion where the pupils, coordinating the table and cartesian representations, individuate the set of the pairs <numbers of lamps, numbers of ribbons>, represent them in the Cartesian plane and generalize the relationship speaking about infinity pairs.)

Teacher**:** (*I go back to the arrow representation*) We must still decide how to read arrows**.**

Pupils cannot find a clear formulation. Then I call in Brioshi[1]**.**
Teacher: Let us try to translate each representation into a message for Brioshi. How did you indicate the number of lamps and that of ribbons in the Cartesian representation?
Almost everybody: with x and y.
We move to the symbolic translation of the correspondence law.
Francesco: I would write 1+2=3, 2+2=4, 3+2=5, 6+2=8
Teacher: What does the arrow mean then?
Finally the pupils answer: +2 (and they write it on the various arrows)
Teacher: If we had 300 firs would we need to write 300 operations?
Everybody: no
Teacher: So?
Davide: x > 2 = y
Matteo: also 3>2 and it is not +2 (*he expresses himself hazily, but he understood what doesn't work in Davide's symbolic writing*)
Silvia: x+2=y
Riccardo: y-2=x
They find out that the last formula translates the inverse arrow: then in the arrow representation they add inverse arrows and on top of each -2
Giulia: 3-1=2
Teacher: using x and y?
Giulia: y-x=2 (*the implicit form appears*)
In the Cartesian plane they close the gaps, adding knots: (4,6) e (5,7)
The transition from qualitative relationships to relationships between numbers is not immediate for pupils. We might insert before this activity a task about approaching relationships between numbers using the arrow representation.
Teacher: What if we wanted to draw the inverse relationship? (*I remind that inverting a relationship means to invert the arrows, hence couples*).
Riccardo goes to the blackboard and marks the knots: (3,1) (4,2) (5,3) (6,8) (corresponding to the couples in figure) and also (6,4) and (7,5). All pupils work on their workbook. We get the graph of the inverse relationship. We compare the two graphs.
... (We avoid to report the part of discussion where the teacher guides the pupils to compare the two graphs from a geometrical point of view)
Teacher: In the formula x+2=y what do x and y represent? *Now I have a new problem. The problem of the name and role of letters. I call up their attention to the couple of points with coordinates (1,3) and (3,1) and on the related functions: x+2=y and y-2=x*
A chorus of pupils: x the first coordinate, y the second
Teacher: So does it work with the pair (1,3)?
Silvia: yes because 1+2=3
Teacher: What about all knots of the same series of points?
Silvia: yes, *she repeats the same operation for all points belonging to the first graph*
Teacher: Shall we consider now the inverse y-2=x? Is it correct if it refers to the knots in the second series?
In the class yes' and no's can be heard.
Annalisa: it does not work because if x is the second it is not right
Teacher: Take the pair (3,1), which number should I substitute for y in the formula?
Valeria: 3 because 3-2=1, if you put 1-2 it does not work
They agree and we write the rule: in the formula y-2=x, y is the first coordinate and x the second.

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[1] *Brioshi* is a hypothetical Japanese boy who does not know Italian but can communicate through mathematical language. Turning to Brioshi facilitates pupils to appropriate the problem of making formal representations and leads them, through forms of *algebraic babbling,* to gradually acquire the correct way to realise them (for deepenings see Malara & Navarra 2001b).

Francesco points out: we invert axes but not in the drawing, only in our head.
For the moment I have not reached the objective: The correct symbolic writing of the inverse function x-2=y. Maybe it is early. I will tackle the problem again as soon as a favourable opportunity turns up. Anyway, pupils have identified the mistake and adapted conventions to their needs.

SOME REFLECTIONS ABOUT THE DISCUSSION

Analysing the classroom episode two are the main elements of discussion: 1) the pupils' difficulty to interpret and codify the arrow representation; 2) the hypostatization of letters.

As to the first point, the core of the question is the conflict between verbal representation (requiring to express the number of ribbons through that of lamps) and arrow representation (that starts with the number of lamps). (We note how pupils, given the difficulties met in verbal representation, change representation register, moving to the Cartesian one.) In this case the process of making explicit arithmetic links between components of the pairs had to be guided. The individuation of the meaning of the arrow and of the verbal representation of relationship is very delicate because two obstacles overlap: the relationship involve the same kind of variables (numbers) and has to be expressed through a sentence of equality. The scheme

$$\boxed{\text{number}} \xrightarrow{\boxed{\mathbf{+2}}} \boxed{\text{number}}$$

induces a sequential reading "the number … increased by 2 *gives* the number …", but it has to be converted in relational terms: "the number … increased by 2 *is equal to* the number …" . Only through this conversion the pupils can enact the arithmetical writings 1 + **2** = 3; 2 + **2** = 4 ; 3 + **2** = 5 which allow them to see them in general terms and arrive at the algebraic formulation of the relationship.

In the discussion made with the teacher she claimed to be convinced that it is appropriate to leave pupils as free as possible in exploration, but she ended up considering her own intervention to be of little influence, understanding that she needed to encourage pupils to make these links explicit.

As to the question of the hypostatization of letters, we note that the teacher immediately refers to variables y and x (this reveals a certain rigidity in her mental model on the representation of functions).

We discussed with the teacher about the fact that if drawing on standard letters pupils have a lower possibility of elaborating, through the examination of particular cases, the abstract meaning of the reference frame O(x,y) and how this reference frame has already been assumed by pupils, as can be seen in various moments of the class episode (pupils labelling of Cartesian axes; the type of key introduced**;** the shared acceptance of names x and y for the various pairs' components). We also highlighted the problematic effects of her initial choice of referring to letters x and y in indicating the two variables. This is mirrored in the conflict between algebraic and Cartesian representations for the need to swap x and y in the algebraic representation of the inverse when referred to the frame O(x,y). Francesco's sentence is meaningful: "*we invert axes, but not in the drawing, only in our head*" but it does not have any follow up in the class.

The collective exchange brought the teacher to grasp the limitations of her choice and to value the use of various letters. In other experimentations, verbal formulations of direct and inverse relationships were made explicit through the use of other letters to indicate variables (linked to their names or to the names of sets of numerical values

assumed by each of them). In enacting the Cartesian representation this lead to labelling the axes consistently with the names of variables and to avoid the obstacle we described earlier. In some cases the reference O(x,y) is introduced by some teachers as a *neutral* frame, useful for comparing the two graphs.
As to the pupils we can say that they have faced with flexibility the work around the functional relationships and surely have learnt to activate different representations. The translation between different representational registers appears more problematic and requires not only collective activities but also individual work.
Let us now reflect on this and other analogous experiences in a wider sense.
Many cases we have analysed in which teachers show that they do not preview pupils' possible answers, grasp a pupil's reasoning or fail to give due value and let drop significant contributions, or are conditioned by some pupils' invasiveness, or are even unable to use appropriate silent pauses (see Malara et. Al. 2004) – clearly show the importance of a *fine* teachers education on orchestrating discussions and mainly on previewing their possible developments as to their ways to formulate questions and to react to the pupils' answers.
This condition poses to us the hard challenge of how to best help the teachers to "fine-tune their antennas" and acquire that "local flexibility" which enables them to adapt to the flux of thoughts which emerges from the class, to grasp the potentialities, to develop them and adequately insert them into the working context. The task is far from being easy, since it is not a matter of dialogue on a mathematical knowledge, but on the more complex and delicate level of behaviour – mostly subconscious – that is rooted in the teacher's past life experiences.
These experiences have made us aware of the fact that we have to implement even finer modalities, to encourage teachers to reflect upon their own actions, thus acquiring new abilities towards "knowing-to-act in the moment" (Mason & Spence, 1999). For example, we deem it indispensable to make use of tools such as video recordings of class interventions (up until now only marginally used in Italian research for question of the 'privacy-law'), to help teachers to analyse the use and incidence of non-verbal language.
A further, completely different, and important ground for reflection is for us the incidence of the network of socio-emotional relationships within the classroom (leaderships, power groups, median roles, singles) in the development of discussions. In many cases, we observed rivalries between groups of different sexes, complicities between singles, or even a refusal on the part of pupils to have themselves involved. These observations have brought us to consider the pupils' participation as an important question of research. We currently promote among the teachers the study of this issue (Nasi 2006).

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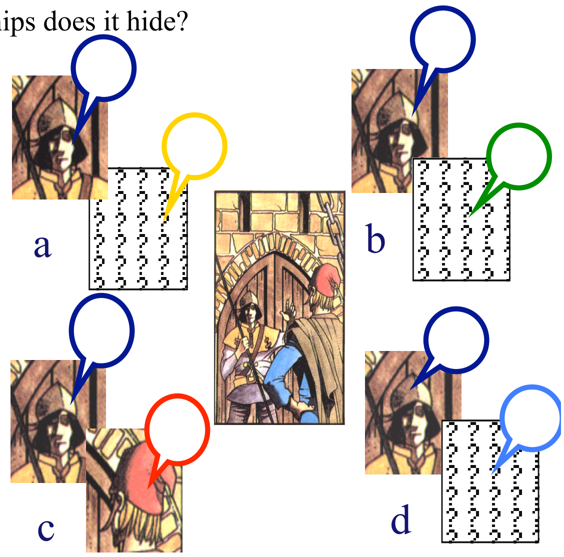
Appendix

Situation 1A. ***A masked ball and a special ... password***

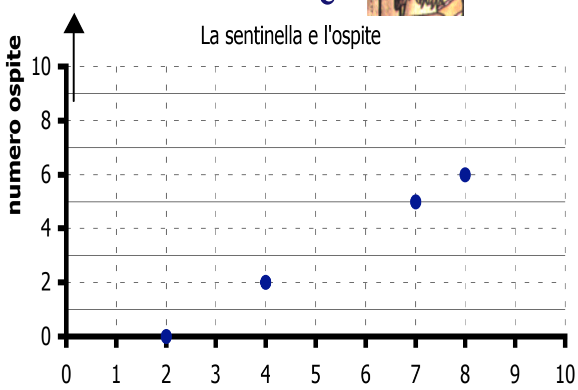
- How would you describe this situation? which relationships does it hide?
- Represent them with the arrow language.
- What happens if you invert che arrows?
- The guard says 231. What does the guest answer?
- A guest answers 450. Why?
- What would you write on the invitation card for this
- party?
- Challenge: and with letters?



Situation 1B. ***Other points of view***

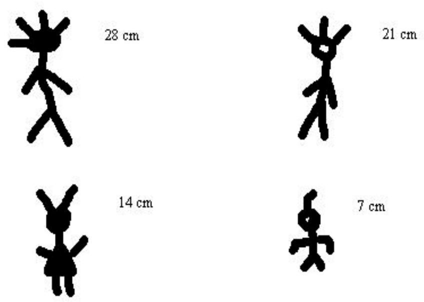
Francesca, who is tired of arrows, describes the relationship as follows:

nr. guard	nr. guest
2	?
3	
4	2
5	
6	
7	5
8	6
…	
15	
…	…



Do you agree with her?
• Would you be able to complete the table and the diagram?
• Would you be able to represent the inverse relationship in the same way?

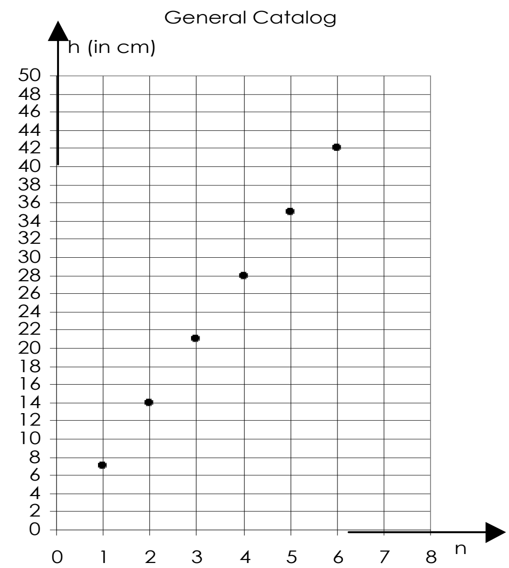
Situation 2A: *Engravings*
On the Aral mount , in the middle of the desert,
the archeologist Giancarlo found images engraved it the rocks.
He reproduced them on his notebook, taking note of their height.
Here is the page:



His team intensely discusses about the possible hidden relationship of the graffiti. In particular
Nicola says: “You just have to multiply by 7 the number of horns on the head; so you can find out the height of each engraving”.
Battista says: “Well, it is clear that if you divide by 7 the heigt of the engravings you find out the number of horns.”
And Paolo: “Waht are you saying! The number of horns is given by the height divided by 7 !”
Ernesto speaks, but the radio covers part of his words :
“I would say that the relationship between zzzzzzzzzzzzzzzzzzzzz remains zzzzzzzzzzzz”.

- What do you think about it?

Situation 2B: *An interesting cue*
Giancarlo carried out some researches in the library and discovered that some years before a group of british archeologists had already analised the engravings of the desert rock. Their paper, published on a prestigious international review, included this diagram:



Even if Giancarlo doesn’t speak much English, by observing it carefully Giancarlo deduced that he must go back to the desert, in order to look for other engravings (in the hope that they still exist).
He already knows how high they shoud be and how many horns they should have.

- How did Giancarlo draw these conclusions?
- What do the following pictures look like?