Unit 1

Brioshi
and the approach to algebraic code

U1. Brioshi and the approach to algebraic code

1. The Unit

Brioshi is a character born in 1998 in a first year, intermediate school, included in the ArAl project. He is a Japanese 'virtual' pupil, whose age varies according to the age of his interlocutors. He doesn't know Italian, but he can express himself in a correct mathematical **language**. Brioshi's class loves meeting other non-Japanese pupils of the same age to exchange mathematical problems. The messages, especially those involving younger pupils, can also contain sentences written in Italian (or even in Japanese, which amazes the class). However, they accomplish an important affective function, like: "Dear Brioshi, we want to know if you can … "; the core of the messages is anyway represented by the mathematical part of the message itself.
For instance, a third or fourth year, primary school can write:

"Dear Brioshi, let's see if you can **solve** this problem: 33 = a - 26"

Brioshi has more than a possibility, he can answer:

$$33 = 59 - 26$$

But it can also give a more complex answer, like:

$$a = 59 \qquad \text{or:} \qquad a = 33 + 26$$

Every time it is clear that Brioshi's answer itself becomes a problem of interpretation for the class who sent the message. Moreover, it is Brioshi's answer, which opens new spaces to be explored.
If Brioshi gives a wrong answer, like

$$a = 60$$

the class can, in their turn, elaborate an answer like:

$$a \neq 60.$$

2. Didactic aspects

Brioshi is introduced in every class taking part in the project, as it represents a very powerful **mediator** used for the transmission of an important but difficult concept that has to be understood by pupils aged between 8 and 14: the need for respect of rules when using a **language**. This need is even stronger when using a **formalized** language, because of the extreme conciseness of the symbols employed. In this sense, Brioshi represents a very powerful **metaphor** in order to emphasize the **process** and in order to reflect on some **semantic** and **syntactic** aspects of the language.

3. General aspects

Brioshi can be presented in two ways:
(i) in a *natural* way, as soon as the opportunity arises; generally this happens if the teacher has the working habit of paying particular attention to the linguistic aspects of mathematics, and so he puts his class in front of matters such as **representation** and **translation;**
(ii) in a programmed way, in a particular (and suitable) moment during the didactic activity, when the teacher starts facing mathematics under a linguistic point of view.
In any case, it must be pointed out that Brioshi:

- is a Japanese pupil;
- *doesn't* know Italian;
- can communicate with us *only* through a mathematical language;
- can interpret mathematical language;
- enjoys himself proposing and solving problems;

According to the existing conditions (age of pupils, teacher's working style) Brioshi can be introduced explaining that:

- there is the chance of exchanging messages containing mathematical problems to be solved with a Japanese pupil;
- it has been agreed with Brioshi's teacher to start with some simple translations;
- the exchange will begin with a simple sentence in Italian, that every pupil will try to translate into mathematical language;
- the different translations will be copied on the blackboard and then discussed **collectively** in order to choose the one to send to Brioshi (remember that Brioshi doesn't understand a single word of Italian);
- once the translation has been sent, pupils will wait for Brioshi's *answer* and will then interpret it;
- Brioshi's class, too, will send problems. Once *interpreted*, the problems will be *sent back*.

There is more than one way to carry out the activity (once again depending on the environmental conditions).
(i) The exchange can be simulated within the class itself; at the beginning the teacher will propose Brioshi's 'return messages', or some pupils might be asked to propose them. This 'role play' always works, whatever the pupils' age;
(ii) The exchange becomes more interesting and efficacious if it occurs between two classes of the same school, or of two different schools, through an exchange of messages.

(iii) Other very effective devices are also messages brought in class by the teacher himself, as if he had received them personally (in a traditional way or by e-mail). These messages stimulate the class's curiosity and represent a challenge for the pupils' logic-mathematical abilities.
(iv) The modality which is the most interesting under many aspects is an exchange of messages between the two Brioshi-classes engaged in a 'mathematical communication' in real time through a sort of specific chat-line. At present (2001-2002) this type of exchanges is being experimented using the *MSN Messenger Service* software.

4. Note

The fact that most Diaries presented in the unit regard classes between third and fifth year of primary school (8 – 10 year olds, apart from the enclosure in which a second year of intermediate school appears) should not cause any misunderstanding; this choice was meant in order to highlight the possibility– and opportunity – of activating (in an arithmetic environment) an approach to **pre-algebraic thought** and to algebra as a language since the first years of primary school. As a matter of fact the Brioshi project is structured not only as a separate Unit (although it possesses all the necessary features), but rather as a *transversal* activity in relation (i) to the other Units of the ArAl Project, in which there are several references to it; (ii) to mathematical languages; (iii) to the subjects of study; (iv) to the pupils' age. As soon as the teacher decides to approach to what has been defined as **algebraic babbling**, Brioshi can appear, and be 'introduced' to the pupils of Second year of Primary school (7 year olds) as well as to the students of a First year, intermediate school (11 year olds), maintaining its power as a didactic mediator also with the 'grown-ups' of third year intermediate school (13 year olds: «You don't know how to write the formula? Try to imagine how you would translate the problem for Brioshi)». To conclude: the Brioshi Project is intended to represent as the instrument for an innovative didactic methodology through the indication of appropriately articulated situations, and as a hint to teachers for a reflection on his mathematical knowledge. Moreover, it may be useful in particular for a revision of his concepts (his *personal epistemology,* so to speak) regarding the *links* and the *oppositions* between arithmetics and elementary algebra and their didactics.

5. Terminology and symbols

Phase	Sequence of situations of growing difficulty referred to the same subject.
Situation	Problem around which individual, group or class activities are developed.
Expansion	Hypothesis of work on a possible expansion of the activity towards an algebraic direction. Its realisation depends on the environmental conditions and on the teacher's objectives.
Supplementary activity	Enlargement towards subjects related to those developed in the preceding Situations.
Note	Methodological or operational suggestions for the teacher.
	In the square a problematic situation is proposed. The text is purely indicative; it can also be presented as it is, but generally its formulation represents the outcome of a social mediation between the teacher and the class
Representation	An underlined word in boldface type highlights a link to a subject illustrated in the Glossary.

Square containing the outline of a typical **discussion**; the following symbols may appear:

√	Intervention of the teacher
	Intervention of a pupil
	Summary of several interventions
	Summary of a collective discussion (a principle, a rule, a conclusion, an observation, …)

6. Phases, situations and subjects

All the phases are abundantly supported by significant extracts from the **Diaries** of activities in the classes.

PHASES	**SITUATIONS**	**SUBJECTS**
First	**1 - 3**	Activities of translation from natural language to arithmetic language of sentences like “to 4 subtract 2” (individuation of equivalent sentences or not comparison between **paraphrases**).
		Scheme of interpretation of the translations proposed by the pupils.
Second	**4 - 9**	Translations connected to the ‘game of the hidden number (“I think of a number, if I subtract 3, I obtain 8. What is the number?”). Meeting Brioshi.
Third	**10 - 11**	Activity on the opposite process: interpretation and translation of written **sentences** into mathematical language.
Fourth	**12 - 14**	Cross translation game (two versions). Diaries of activities in the classes.
Fifth	**15**	More complex problems with a missing datum to be translated with a letter.
Attachments		Complete diaries of activity in the classes. Diary of the dialogue between two Brioshi-classes using the *MSN Messenger Service* software.

7. Distribution of situations in relation to the pupils’ ages

The distribution represents an indicative proposal based on the experience in the classes taking part in the project. We repeat that the Brioshi Project becomes really significant when introduced *transversally* with the other didactic activities and when as a constant routine the *linguistic aspects* of mathematics are highlighted.

			PHASES AND SITUATIONS														
			I			**II**						**III**		**IV**			**V**
School	Age	Cl	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**
Primary	**7**	**2**	12 hours														
	8	**3**	15 hours														
	9	**4**	15 hours														
	10	**5**	15 hours														
Intermediate	**11**	**1**	15 hours														

Suitable age related activities	*6*	***7***	***8***	***9***	***10***	***11***	*12*	*13*	***Comments***

First phase [1]

1. Brioshi is introduced in ways similar to those described.

Note 1

The class is given tasks concerning the four operations (see the bottom of the page); at first the pupils interpret orally their meanings and if they want they execute mentally the relative calculations. As soon as possible, they pass to a written translation of the tasks, which have been interpreted as problems to be translated for Brioshi.

The lists contain some paraphrases [2]; the purpose here is to make pupils reflect on a very delicate aspect connected to mathematical **writing**: the **relationship** between meaning and denotation. Let's make this concept clear.

A mathematical object can be denoted in many ways, each one with a very different meaning. If we consider two equivalent denotations like '14: 7 + 2' e '4', the second (which might be seen as the '**result**' in relation to the first) is more opaque because the aspect of process has been lost, while that of result is privileged.

What is proposed to the class is then the comparison between different **codifications** of a same process; every instruction can be read at different levels, also according to the way in which its formulation in natural language is organised. For instance:

(i) *'Add 12 to 2'* favours the **transparency** of the *process* and leads to a translation like: '2 + 12';

(ii) *'Add 12 to 2 and give the result'* favours both transparency of the *process* and individualization of the *result* and leads to a translation like: '2 + 12 = 14';

(iii) *'Find the sum of 2 and 12'* leads to execute a calculation, and thus to find a *result*, of course only after identifying what is the process involved (for example, a pupils might not recognise the meaning of 'sum'). Both '2 + 12 = 14' and '14' are acceptable translations; the discussion will point out the differences between the two representations and will lead to significant conclusions regarding the suitability of one rather than the other in relation to a given objective. The unit contains several examples of this kind of discussion.

The activity proposed in this first phase is thus introductory to the real exchange of messages with Brioshi, and directs the pupils to a reflection on the mathematical language and its conscious use [3].

Examples of tasks are now presented, which refer back to what has been said in (i), (ii) e (iii). The teacher can choose among them, or he can draw inspiration for similar tasks to be created. For convenience, the tasks are divided on the basis of the operations they refer to; the 'dimension' of the numbers involved is merely indicative.

In the comments the related examples of correct translations can be found [4].

2.

(i) "Translate the following sentences into mathematical language" (the clarification of the process is promoted).

• Add 5 to 7	• Subtract 8 from 15
• 8 add 3	• 13 take away to 17
• Add 11 to 6	• 19 subtract 3
• 9 plus 4	• 2 less then 11
• Add 7 to 24	• Double 4

(Continues in the following page)

[1] *With the help of a great variety of paraphrases, the aim is that of favouring control on the mathematical meaning of sentences expressed in natural language.*
The numbers involved vary according to the age of the pupils, but it is anyway important that they are not too big, because the main objective is not mental calculation, but a stimulation to grasp the meaning of the writing and the control of its translation into mathematical language. Too big numbers might inappropriately remove the attention (and worries) on calculation rather than on interpretation.

[2] *We don't consider as necessary to respect a particular order in proposing tasks. Some might seem very simple to translate, but the experiments show that it is not always so.*

[3] *The activities described need to be integrated in the normal didactic work, in order to favour a real competence in the passage from natural to mathematical language and vice versa.*

[4] *In sentences like "add 8 to 3" the assumption that the* commutative *property is valid, must be considered as implicit. Anyway, it is better to draw the pupils' attention on the fact that, for example, in "Add 11 to 6", the translation "11 + 6" does not express the 'literal' meaning of the process, but it turns it into an equivalent one, induced by the text.*
These aspects will be examined later in the 'Scheme of interpretation of translations'.

2.

(i)
7 + 5
8 + 3
6 + 11
9 + 4
24 + 7

15 – 8
17 – 13
19 – 3
11 – 2
4 + 4 (of course also 4 x 2)

Suitable age related activities	6	7	8	9	10	11	12	13	***Comments***

→
- Multiply 5 by 9
- 4 times 7
- Double 6
- Divide 24 by 3
- Share 42 between 2
- Halve 14

(ii) 'Translate the following instructions into mathematical language' (we can underline both the process and the **product**)

- Add 9 to 14 and indicate the result
- Add 10 to 8 and indicate the sum
- Triple 5 and find the result
- Take away 7 from 12 and show the difference
- Take away 5 from 11 and indicate the result
- Subtract 8 from 19 and find the difference
- Multiply 5 by 9 and indicate the product
- Multiply 7 by 4 and find the result
- Divide 24 by 3 and show the quotient
- Divide 56 by 8 and indicate the result
- Halve 10 and write the result

(iii) "Translate the following sentences into mathematical language" (the instruction emphasizes the result, but it contains an 'implicit invitation' to represent the process, too).

- What is the result of the addition between 2 and 12?
- Find the sum of 14 and 5
- How much is 13 and 9?
- Find the result of the subtraction 15 minus 9.
- Find the difference between 11 and 7.
- How much is 24 minus 16?
- What is the difference between 18 and 8?
- How much is 7 smaller than 13?
- Find the product of 8 and 6.
- What is the result of the multiplication of 9 and 3?
- Find the double of 11.
- How much is 5 times 8?
- How much is 6 multiplied by 5?
- Find the quotient between 32 and 4.
- What is the result of the division between 72 and 8?
- Find the half of 30.
- Find how many times 10 is contained in 40.
- How much is 63 divided by 7?

Comments:

5×9
4×7
$6 + 6$; 6×2

$24 : 3$
$42 : 2$
$14 : 2$

(ii)

$14 + 9 = 23$
$8 + 10 = 18$
$5 + 5 + 5 = 15$ ($5 \times 3 = 15$)

$12 - 7 = 5$
$11 - 5 = 6$
$19 - 8 = 11$

$5 \times 9 = 45$
$4 \times 7 = 28$

$24 : 3 = 8$
$56 : 8 = 7$
$10 : 2 = 5$

(iii)

14 $2 + 12 = 14$
19 $14 + 5 = 19$
22 $13 + 9 = 22$

6 $15 - 9 = 6$
4 $11 - 7 = 4$
8 $24 - 16 = 8$
10 $18 - 8 = 10$
6 $13 - 7 = 6$ $7 + 6 = 13$

48 $8 \times 6 = 48$
27 $9 \times 3 = 27$
22 $11 \times 2 = 22$
40 $5 \times 8 = 40$
30 $6 \times 5 = 30$

8 $32 : 4 = 8$
9 $72 : 8 = 9$
15 $30 : 2 = 15$
4 $40 : 10 = 4$
9 $63 : 7 = 9$

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Suitable age related activities	6	7	8	9	10	11	12	13	Comments

As anticipated in the presentation, the unit is organised in a sequence of significant fragments of diaries of the activities carried out in the classes, through which (i) *method* and (ii) *strategy* are exemplified.
(i) From the point of view of *method*, the role played by the collective discussion on the translations proposed by the pupils will be evident. The aim of this activity is to favour a deep reflection on their meaning; in other words to stimulate, through an intensive metalinguistic activity, the development of metacognitive attitudes.
(ii) From the point of view of *strategy*, it must be highlighted how Brioshi can represent a powerful metaphor to train pupils both to the use of a coherently structured language and to the ability of interpreting (and manipulating) sentences expressed through mathematical symbols.
The activity proposed in this Unit is potentially very resourceful, but initially not easy to handle because it requests a progressive refining of the ability to (i) setting up an acute analysis of the pupils' **protocols**, (ii) recognising similarities and differences between writings, which is often not at all trivial and finally (iii) leading the discussion in a productive way. More generally this implies an extension of the teacher's sensitivity towards the linguistic aspects of mathematics, so that, even recurring to the Brioshi metaphor becomes a widespread and consolidated habit.
In order to help in this process, we propose (see following page) a *Scheme of interpretation* of the pupils' writings seen as translations, exactly as it happens in linguistics between two different natural languages. Consequently, both the organization criteria for the scheme and the terminology employed contribute to consolidate an idea of mathematics in a linguistic perspective.
Before going on with the **Situations**, we present an example of a significant summary of an activity Diary in class.
Diary 1 (primary school, 9 year olds, December).

Translate the sentence for Brioshi: Subtract 8 from 15

Translations written on the blackboard and submitted to discussion:
L [5]: (a) 15 – 8;
LN: (b) 15 – 8 = 7
LO: (c) 15 – 8 =

The first interventions highlight the difficulty of comparing and interpreting the writings; the similarities between writings are not grasped.
Most pupils consider as positive the introduction of the symbol '='. The (a) is considered as 'incomplete'.
Lucia notices that in her opinion the most correct sentence is (a) (hers) «… because it is not requested to find the result but to translate the sentence in a way that Brioshi can understand» [6,7].
The remark is not understood by many classmates. The class cannot choose a writing.
✓ We ask to write number 4 in other ways.
Proposals: 1 + 3, 2 + 2, 5 – 1, 20 – 16, 2 × 2, 4 × 1, 100 – 96.
✓ We asked why they never proposed to write the symbol '='.
«Because you asked us something that gives 4 as a result [8]».
«You asked other ways to do it, *not another operation* [9]».

[5] *The bold typed symbols are described in the Scheme of the following page.*

[6] *So far, the discussion seems to be rather pointless. Nonetheless, there are two interesting aspects.*
- *The* ***equal*** *appears in every writing except one; it is seen by the pupils as an* ***concluding indicator*** *and it expresses the idea that sooner or later it must be reached. It denotes the prevalence of a leading* operative *attitude. On the other hand, the absence of the symbol is seen as* ***'lack of closing'*** *for the operation (and in fact the class considers sentence (a) as 'incomplete').*
- *Lucia's remark explains in a simple way, why the equal is not necessary and opens the way towards the individualization of the difference between solving and representing a problem.*

[7] *Often the pupils ask if "they have to write the result". We give here an example of how we tried to clarify the concept of "translating" - "solving" recurring to the English language.*
We ask how to translate the sentence "Hai una matita?" The class answers «Have you got a pencil?» The pupil we are addressing really has a pencil in his hand and he answers «Yes». The class understands that in the first case the question has been translated*, while in the second case an* answer *has been given.*

[8] *The use of the verb "give" ("fare" in Italian) induces the idea that the proposed writings are seen as different ways of writing the result 4 (as it is often said in common language "qualcosa che fa 4"), rather than "equivalent to 4".*

[9] *This conclusion is the confirmation of what had been stated before in relation to the stereotype that the equal sign 'prepares' the individualization of the result (we might say: the conclusion of a* story *that evolves in time).*

Scheme for interpretation of the translations proposed by the pupils

N.B. (a) all the entries in the scheme are supplied with examples of protocols; (b) the letters **L**, **F**, **S**, etc. appear in the diaries beside every translation; (c) a translation can have more than a feature.

I THE TRANSLATION SHOWS A GOOD COMPREHENSION OF THE SITUATION:

L **Literal**: *use of the letter* (if necessary); translation adherent to text.
The pupil understands both the problem and the instructions. It expresses a good command both on a cognitive level and on a metacognitive one. The use of the letter is explained with sentences like "p means the place of the number I don't know" (third year, primary school, 9 year olds), "n represent the missing number" (third year, primary school, 9 year olds). In these cases the translation is always **relational**; sometimes the writing is correct but 'exceeding' the text (**Le**). It can show the *inversion* of the addends or of the factors (see examples). In many cases the translation can be **'free'** (**Lf**): the pupil guesses a correct connection among the numbers; the translation witnesses an elaboration of the verbal instructions (see examples).
"How much do you need for 6 to get to 9" *Transl.* ***L****: 6 + a = 9* ***Lf****: 9 – n = 6*
"From 15 subtract 8" *Transl.* ***Le****: 15 – 8 = p*

F **Faithful**: *use of symbols other than letters*, translation adherent to text.
The pupil understands the logical structure of the problem but performs a partial metacognitive command on the meaning of the symbols he uses; sometimes the writing is correct but '**exceeding'** the text (**Fe**, second example); often it is written 'in a column' showing a leading operative attitude (**O**, see last entry).
"I think of a number, add 4 and obtain 10"
Transl. ***F****:* *+ 4 = 10* *... + 4 = 10? + 4 = 10* *□ + 4 = 10* *✷ + 4 = 10*
"From 15 subtract 8" *Transl.* ***Fe****: 15 – 8 = ✷*

S **Sense-oriented**: a leading operative attitude emerges, *translation adherent to text*.
The pupil expresses a general comprehension of the logical structure of the problem but performs a partial metacognitive command; the translation 'prepares to the result' and prearranges for a **directional** rather than relational reading.
"I think of a number, add 4 and obtain 10" *Transl.* ***S****: 10 - 4*

II THE TRANSLATION SHOWS A PARTIAL COMPREHENSION OF THE SITUATION:

C **Confused**: *partial adherence of the translation to the text.*
The pupil uses, links or elaborates in an inappropriate way the starting data; the language is often mixed (natural/symbolic).
"How much do you need for 6 to get to 9" *Transl.* ***C****: 6 ? 9*

III THE TRANSLATION SHOWS AN INSUFFICIENT COMPREHENSION OF THE SITUATION:

I **Inaccurate**: *lack of adherence of the translation to the text.*
The pupil links the starting data in a wrong way (often with a wrong operation).
"I think of a number, add 4 and obtain 10" *Transl.* ***I****: 10 : 4*
"How much do you need for 6 to get to 9" *Transl.* ***I****: 6 + 9*
"I think of a number, add 4 and obtain 10" *Transl.* ***I****: 4 E 10+ 4 =* *+ 4 – 10*
... 10 - 4 = ... *14 – 4 = 10*

A **Adulterated**: *total lack of adherence of the translation to the text.*
The pupil shows a lack of comprehension on every level (it can derive from a temporary misunderstanding, wrong reading of the text, lack of comprehension of the **didactical contract**). In the case of more articulated problems, the translation can consist of an iconic representation of the problematic situation, occasionally faithful, but 'blocked'.
"I think of a number, add 4 and obtain 10"
Transl. ***A****:* *1 + 13* *7 – 10* *10 × 10* *9 + ... = 19*

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N **Resolutive numerical**: *insertion of the value of the* ***unknown***, scarce adherence of the translation to the text.
The pupil understands the situation partially and performs a poor metacognitive command; he performs the operation without realising that writing does not represent a 'problem' (short-circuiting of the problem).
"I think of a number, add 4 and obtain 10" *Transl.* ***N****: 6 + 4 = 10* *... = 6 – 10* *6 + ... = 10* *6 + 10*

O **Operative***: insertion of the '=',* adherence of the translation to the text, any kind of writing may appear.
The pupil shows a preference for 'calculation' rather than for 'representation'. The lack of '=' is seen as a 'lack of closing'. The writing can be both 'in line' and 'in column'.
"I think of a number, add 4 and obtain 10" *Transl.* ***O*** *:* *6 + 4 =*
"From 15 subtract 8" *Transl.* ***O****: 15 – 8 =* ***FO****: 15 – 8 = ✷*

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Suitable age related activities	6	7	8	9	10	11	12	13	Comments

3. It is advisable to introduce also situations, which are opposite to the ones proposed so far, and make the pupils translate mathematical sentences, stimulating the production of paraphrases and their comparison.
The following diary illustrates an activity in this direction and contains a spontaneous approach to the *use of letters in a mathematical environment*, anticipating the central theme of following situations of the Unit.

Diary 2 (primary school, 9 year olds, December)

Translate the sentence into natural language: 7 – 3
Numerous doubts **10** «4». «Seven minus three». ✓ We ask 'more courage'. These translations are proposed: «From 7 I subtract 3» «From 7 I take away 3» «I take away 3 from 7» «7 diminishes *to* 3» ✓ We correct the last sentence from a linguistic point of view the example of temperature (diminishes *of* …). Unexpectedly a pupil proposes a translation into a problematic form: «If I add a number to 3 and obtain 7, what's the number?» ✓ We ask him to translate the sentence into mathematical language. He writes: 3 + a = 7. ✓ We ask the class what Brioshi's answer might be. «a = 4».

Second phase

4. The pupils must now face instructions, which are more complex from the logical point of view and more articulated from the linguistic one **11**. In the preceding cases the translations were made starting from a couple of numbers (possibly the data were in an implicit form, as in 'double the 6'). On the contrary now, the sentences put the pupils in front of a 'mysterious' or 'missing' number and lead them *to find a way to* represent it. Substantially they are *paraphrases in the natural language of open mathematical sentences* **12**.
The instruction is of the type:

"Translate this sentence so that Brioshi understands that he has to find the mysterious number".

- I add 8 to a number and obtain 18.
- I add 8 to a number and obtain 18. What is the number? **13**

(Continues in the following page)

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10 Even when, as in this case, the instruction is precise, there can be some doubts in the interpretation of the problem. The writing '7 – 3' suggests the operation, *and thus* the search for a *result (the first intervention «4» is not accidental).*
*At first very few pupils grasp the existence of two different points of view: operative and representative. In the first the **computational** aspect prevails, in the second the relational and linguistic aspects prevail. Brioshi helps in the distinction of the two spheres.*

11 The sentences contain also opposite operations, more difficult to translate. We suggest to start from additions *and to pass to subtractions later on.*

*12 As will result clearly from the following diaries, pupils propose a great variety of translations, and in the first writing may not appear a letter. Its use should be the result of a slow '**social** conquer' made by the class.*
A teacher's task will be that of introducing the use of letters in a natural way, limiting external pressures. Our experience tells us that, just like in learning natural language, every class follows its own personal rhythms.
Another important aspect concerns the use of particular letters. If it is true that in algebra there are some conventions, at this school level, any letter fits the purpose.
In the following examples ('standard' translations) the use of letters is only indicative; teachers might well expect a wide range of variants, more or less correct.
For more details see the next Diaries.

Formal codifications:
$n + 8 = 18$
$n + 8 = 18$ *or* $n = 18 - 8$ $\quad n = 10$

13 It is interesting to draw the pupils' attention on the difference between this instruction and the preceding one.

Suitable age related activities	6	7	8	9	10	11	12	13	Comments

- What number do I have to subtract from 20 to obtain 3?
- How much is 11 bigger than 3?
- How much is 4 smaller than 6?
- How much do I have to add to13 to obtain 30?
- I think of a number, subtract 3. The rest is 7. What's the number?
- I multiply 6 by a number and obtain 42. What's the number?
- What number do I have to multiply by 6 to obtain 24?
- How many times is 5 contained in 35?
- A number is contained 7 times in 56. Find the number.
- What number has to be multiplied by 9 to obtain 72?
- How do I divide 63 to obtain 9?
- If I divide 54 by a number I obtain 6. What's the number?
- The difference between two numbers is 4. What can the two numbers be? Find some couples.
- The product of two numbers is 12. What can the two numbers be? Find some couples.
- The quotient between two numbers is 4. What can the two numbers be? Find some couples.[14].

$20 - n = 3$ $\quad n = 20 - 3$
$3 + n = 11$ $\quad n = 11 - 3$
$4 + n = 6$ o $6 - n = 4$; $\quad n = 2$
$13 + n = 30$ $\quad n = 30 - 13 = 17$
$n - 3 = 7$ $\quad n = 7 + 3$
$6 \times n = 42$ $\quad n = 42 : 6 = 7$
$n \times 6 = 24$ $\quad n = 24 : 6 = 4$
$35 : 5 = n$ $\quad n = 7$
$56 : 7 = n$ or $n \times 7 = 56$
$9 \times n = 72$ $\quad n = 72 : 9$
$63 : n = 9$ $\quad n = 63 : 9 = 7$
$54 : n = 6$ $\quad n = 9$
$m - n = 4$
$a \times b = 12$
$a : b = 4$

Three Diaries follow - concerning three different classes - which illustrate the possible developments of these activities in the classes; the translations proposed are classified on the basis of the entries in the ***Scheme for interpretation of the translations proposed by the pupils***.

5. Diary 3 (primary school, 9 year olds, beginning of school year)

Translate the following sentence for Brioshi:
Add 4 to a mysterious number and you'll obtain 10

The pupils' translations written on the blackboard and discussed:
F: (a) + 4 = 10; (b) 4 + = 10 [15]
S: (c)10 - 4
FO: (d) + 4 = 10 =
I: (e) 4 E 10 (f) 4 = 10
IN: (g) 6 + 10
N: (h) 6 + 4 = 10; (i) 6 + 10
A: (i) 1 + 13; (j) 3 + 10; (k) 3 + 7; (l) 10 x 10

The class chooses almost unanimously (a) «because the blank space helps Brioshi to understand what the missing number is».
✓ We underline that the computer does not know how to do the blank space and we ask what we can do about it.
Proposals: (m) p + 4 = 10 [16] (n) ? + 4 = 10; (o) @ + 4 = 10.
The girl who proposed (m) explains that p means «the place of the number».
The class decides to send (m) [17].

[14] *The last three instructions are more complex; they are suitable for older and more expert pupils because the representations request the use of* two *letters and the choice of one of the two numbers. The road to equations with two unknowns is open.*

[15] *The writings (a) and (b) represent the occasion for interesting investigations. They are equivalent from the mathematical point of view (it is better to highlight the commutative property), but (a) more than the other respects the sequential aspect, which is implicit in the instruction. Similar observations can be made for "Multiply 3 by 8" and "Multiply 8 by 3".*

[16] *It is interesting that the letter 'p' appears, although later. Nonetheless, we must underline that it very rare that a letter is proposed spontaneously as in this case by a pupil. Sometimes it happens when pupils have observed older brothers using letters.*

[17] *It is advisable to allow discussions to contribute slowly to the highlighting of differences and analogies between the writings. The choice of what is the best one to send to Brioshi depends on the environmental conditions; at the beginning, for example, the blank space is generally the most 'popular' it seems to be clearer on the fact that «Brioshi has to find out the number». The gradual refining of judgement will bring to a clear choice of the most significant writings.*

Suitable age related activities	6	7	8	9	10	11	12	13	***Comments***

6. Diary 4 [18] (primary school, 9 year olds, November)

> Represent this message in mathematical language for Brioshi:
> I think of a mysterious number, subtract 26 and obtain 59
>
> Translations written on the blackboard and submitted to discussion:
> **L**: (a) p – 26 = 59; (b) x – 26 = 59; (c) $p^{?} – 26 = 59$ [19]
> **F**: (d) - 26 = 59; (e) * - 26 = 59 (on the following line) * = [20]
>
> The class expresses uncertainty on the choice because the translations are all basically correct. The letter ‘p’ for ‘place’ is not clear; some pupils think that whatever symbol is fine, but very few understand that it is a *number*.
> «The symbol * seems to replace anything and this doesn’t help to understand that it replaces a number».
> «(a) is right because last time we imagined that Brioshi would give us ‘p’ as an answer instead of the mysterious number».
> «(c) is fine because it shows clearly that there goes a number».
> The class decides to send (e) [21].

7. Diary 5 (primary school, 9 year olds, December)
The class has already had the opportunity of facing the representation of the unknown with a letter [22].

> Translate the following sentence for Brioshi:
> How much do I need from 2 to get to 7
>
> Translate the following sentence for Brioshi:
> How much do I need from 2 to get to 7
>
> Translations written on the blackboard and submitted to discussion:
> **L**: (a) 2 + p = 7
> **F**: (b) 2 + ✳ = 7 (on the follow. line) ✳ =; (c) 2 + ? = 7; (d) 2 + = 7
> **N**: (e) 2 + 5
> **AO**: (f) 2 ✳ = 5 + 7 = ✳
>
> The classmates ask the author of (f) the reason of his writing which is classified as senseless.
> The boy cannot explain it.
> Translation (e) is considered wrong because it contains the answer; (a) (b), (c), (d) are recognised as correct and equivalent.
> «In my opinion (a) is wrong because it would be “subtract 2 to 7”».
> The class decides to send (a) to Brioshi.

[18] *The pupils are more used to* operating *on numbers rather than analysing the* ***relationships*** *that connect them. Activities like these are meant to help the individualization of the relational aspects between the pieces of information contained in the problem, from an arithmetic point of view.*

[19] *(c) not only translates the message, but it also invites to a further elaboration to find the hidden number.*

[20] *Also (e) presents something more than a simple translation, because it* ***roughly*** *contains Brioshi’s answer.*

[21] *The choice of a symbol must be completely free; in* particular the use of a letter must be the outcome *of a collective* ***negotiation****. For example, in this case, the final decision of the class rewards the writing with an asterisk, rather than the two with a letter (more evolved). These aspects are very important to build semantically significant bases to the construction of algebraic language.*

[22] *When a letter starts to appear in a translation next to other symbols, it is advisable to explain the class that, among the several ways to represent a number (dots, question mark, square, blank space, etc.), the one employing the letter is the preferred one by mathematicians and that is has undergone deep mutations in time.*
However, we observed that even when many pupils still don’t use it spontaneously, the letter is ‘accepted’ by the class.
Nevertheless, it is important that the approach to symbolic language is very slow, and passes through gradual steps, as the result of a social mediation (right and wrong interventions, attempts, corrections, intuitions, comparisons, ***verbalizations****, etc).*

(Continues in the following page)

Suitable age related activities	6	7	8	9	10	11	12	13	Comments

8. Diary 6 (primary school, 9 year olds, January) [23]

Translate for Brioshi the following sentence: What number do I have to subtract from 14 to obtain 10.
«14 – n = 10». ✓ We ask to represent the sentence in a graph and to describe it. $14 \xrightarrow{-n} 10$ «14 minus n gives 10». ✓ We invite them to reflect of the definitions given before. «From 14 I subtract n and obtain 10». ✓ We get them to think of the *name* of the operation. «From 14 I *subtract* n and arrive at 10». ✓ We ask to avoid a 'railway' language like 'leave from' and 'arrive at'. «From 14 I subtract n and obtain 10». ✓ We make them complete the graph and ask them to describe it. $14 \underset{+n}{\overset{-n}{\rightleftarrows}} 10$ They get to the following translations. (a) «10 plus n get to 14» (b) «10 plus n and I obtain 14» (c) «To 10 I add n and obtain 14» (d) «To 10 add up n and obtain 14» [24] ✓ We ask to reflect on the name and the result of the operation. (e) «The sum between 10 and n is 14»

→
The activity points to a coordination of the passage to different representations (the algebraic one, which privileges the relational aspect and the ***arrow*** *one, which highlights the* ***procedural*** *aspect) and to consider the operational processes represented by the graphs in relational terms.*

[23] *The class that had significant experiences with the use of letters in arithmetics in the preceding years are of course favoured. In the other classes it might happen that the pupils meet some initial difficulties in understanding the instructions and in setting up the translation.*
However, even if more slowly, the activity gradually builds up and the language gets finer and finer. The pupils learn the ability to identify a paraphrases more and more correct from the logic-linguistic point of view, along with an improvement of their ability to reflect on the relational aspects of mathematical writings.

[24] *The definitions from (a) to (d) are of a diachronic procedural type (they describe the operation and its execution in time).*
On the other hand (e) is of a relational type; it is more evolved because it presupposes a detachment *on the observer's side from the purely operational aspects. The pupils' attention is not focused on 'what to do' anymore, but on the relationship that links three numbers, one of which is unknown.*

Suitable age related activities	*6*	***7***	***8***	***9***	***10***	***11***	*12*	*13*	***Comments***

Third phase [25]

Once again we favour the *opposite* process, by getting the pupils to interpret a sentence written in mathematical language [26].

9. Continuation of **6. Diary 7** (primary school, 9 year olds, December)

The class decides to send this message:

$* - 26 = 59$ (followed by) $* =$

✓ We ask what Brioshi's answer might be.

With evident difficulties, 4 pupils (out of 14) write: * = 85. Many classmates understand the correctness of this writing.

✓ In order to favour comprehension, we propose a graph containing numbers different from those of the problem and ask to complete it:

$$14 \rightleftarrows 36$$ [27]

The pupils add to the graph: + 22 e – 22.

$$14 \underset{-22}{\overset{+22}{\rightleftarrows}} 36$$

✓ We try to force them, by cancelling 22 and – 22 and write: + a and – a:

$$14 \underset{-a}{\overset{+a}{\rightleftarrows}} 36$$

then we ask to interpret the graph again.

«a is number 22»

«a is a number»

✓ We ask to translate the last graph into a message for Brioshi.

The pupils propose:

(a) 14 + a = 36

(b) 36 – a = 14

✓ We propose to translate the messages into problems for Brioshi.

Proposals:

(a) «What number must we add to 14 to obtain 36?»

(b) «What number must we subtract from 36 to obtain 14?»

We end up collectively paraphrasing the initial problem:

«To what number must we subtract 26 to obtain 59?»

[25] *We show here the diary regarding a starting situation containing a* subtraction *(which is more complex than one containing an addition) to highlight some of the difficulties that might arise in class and the strategies employed to overcome them.*
The main obstacle is represented by the fact that Brioshi's answer can only be found by applying the opposite operation 59 + 26.

[26] *All the activities like this, that force to change one's point of view and to 'move' among the different representations, favour the formation of an open pre-algebraic mental habit and to the comprehension of the meaning of writings expressed in a formalised language.*

[27] *The so-called 'arrow' representations favour the comprehension of the direct relationship – opposite operation. As will result more clearly later on, this representation turns out to be really powerful also in the passage to the algebraic field. For instance, consider the instruction: "Brioshi sent this problem:*

n - 17 = 32

find the value of n".
The possible use of an arrow representation favours the individualisation of the formula to be used in order to solve Brioshi's problem:

$$n \underset{+17}{\overset{-17}{\rightleftarrows}} 32$$

32 + 17 = n.

Suitable age related activities	6	7	8	9	10	11	12	13	Comments

10. We put ourselves in Brioshi's shoes and we answer his messages. [28,29]
The activity is illustrated through a comparison between the **Diaries 8** and **9** of two classes (A and B, primary school, 9 year olds) dealing with the problem:

Brioshi has to find out what number he must add to 4 to obtain 10.

Class A *has just begun to use the letter instead of the missing number* and proposes to send this message (**L**):

$$p + 4 = 10$$

✓ We ask what Brioshi's answer might be.

The class writes: 6 + 4 = 10.

✓ We try to force the situation and ask how they would interpret the answer: p = 6

«Brioshi says what number he would write instead of the one he has to find».

✓ As a test, we propose a message by Brioshi and ask them to translate it into natural language and to elaborate an answer:

$$p - 13 = 4$$

«I think of a number, subtract 13 and obtain 4. What's the number?»

$$p = 17$$

Class B *has not used the letter yet and decides to send this message*:

+ 4 = 10 [30]

✓ We ask what Brioshi's answer might be.

The class gives three solutions (a), (b) of a type **N**, (c) of a type **F**:

(a) 6 + 4 = 10 (b) (6) + 4 = 10 (c) + 4 = 10 6

and decides to send (a): 6 + 4 = 10.

✓ We force the situation and imagine that Brioshi sends this problem:

$$n – 2 = 14$$

A pupil elaborates this answer: 16 - 2 = 14

✓ We ask if this is the only possible answer. We don't receive relevant answers. We ask what Brioshi might mean by n.

«*n represents the missing number*».

The pupils cannot elaborate the statement of their classmate.

✓ We ask to invent a problem and to send it to Brioshi using his same language.

«I think of a number, subtract 4 and obtain 19».

The message is translated into: n – 4 = 19

Imagining to receive the answer, we write: n = 23

The pupils consider it wrong and propose: 23 – 4 = 19

They think it over and propose «n, that is the missing number 23».

✓ We pass on to a test and imagine we receive this message to which we must give an answer:

$$36 – n = 28$$

The pupils propose four answers:

(d) number 8 (e) n8 (f) n = 8 (g) 36 – 8 = 28

They choose to send (f): n = 8

Comments

[28] *This activity confirms the effectiveness of Brioshi as an instrument of mediation to introduce the letter instead of the unknown number. Although they know very well that in this moment Brioshi is the teacher himself, the situation is very probable and involves in a sort of stimulating 'role play'.*

[29] *It is important that in the course of activities* the initial hints are alternated *and that every time the activity starts from a problem proposed:*
- *by the teacher for the class in natural language or in mathematical language;*
- *by the pupils for Brioshi in natural language or in mathematical language;*
- *by Brioshi's class for the pupils.*

In the following Situations similar activities will be proposed.

[30] *The starting writing of this class is more 'backward' than that of the preceding class. This solved down the conquer of the letter, but in the end the results are anyway satisfying.*

Suitable age related activities	6	7	8	9	10	11	12	13	Comments

Fourth phase

11. GAME OF CROSS TRANSLATIONS (first version)
Some pairs are formed; in every pair a pupil writes a sentence in mathematical language (for example "9 × n = 27") and the other in natural language (for example "I add a number to 3 and obtain 17; What's the number?"). Then the pupils in every couple exchange the texts, translate them and verify its correctness [31]. Later on they swap roles. The most interesting and controversial matters can become object of a collective discussion.
The pupils are invited to use all the known operations.
Some examples (third year, primary school, 9 year olds); on the left are the original texts and on the right the correspondent translations.

Correct:
(A) "From 5 I subtract d and what remain is 1" → (B) "5 – d = 1"
(B) "19 × n = 81" → (A) "I multiply 19 by a number n and obtain 81"

Correct, interesting, subject to discussion:
(C) "To 4 I subtract 7" → (D) "4 – 7 = A" [32]
(D) "U + 3 = 8" → (C) "U plus three equals eight" [33]

Incorrect:
(E) "To 7 I subtract 4" → (F) "7 + 4 = N"
(F) "N + 1 = 10" → (E) "From 1 I subtract 10"

Incomprehension:
(G) "Nine I obtain a wrong number = 4" → (H) "?"

12. Given an equation, the class invents a problem to send to Brioshi [34]

Diary 10 (primary school, 9 year olds, February)

> Invent a problem that can be translated for Brioshi with the sentence: 30 – n = 6.
>
> Translations proposed by the pupils:
> (a) «I have 30 candies. I subtract n candies and obtain 6»
> ✓ We ask to improve the text linguistically.
> (b) «I have 30 candies. I subtract the missing number and obtain 6»
> ✓ We invite not to use words in 'Mathematikese' (they smile).
> (c) «I have 30 candies. I eat *a few* and obtain 6 candies [35]»
> (d) «I have 30 candies. I eat a few and 6 candies *remain*»
> ✓ We ask what other words can replace 'a few''.
> (e) «I have 30 candies. I eat *some* and 6 remain »
> (f) «I have 30 candies. I eat *some of them* and 6 candies remain»
> We propose another problematic situation:
>
> Invent a problem that can be translated for Brioshi with the sentence: 24 : n = 4. Don't talk about candies anymore, but about hamsters [36].
>
> Translations proposed by the pupils:
> (g) «I have 24 hamsters. I divide them… »
> (h) «I have 24 hamsters and I have some children. I give 4 … »
> (i) «I have 24 hamsters. I give them away and I have 4 hamsters»
> (j) «I have 24 hamsters and some children and I give away 4»
> (k) «I have 24 hamsters. I divide them by four into equivalent groups»
> After many doubts a girl's face enlightened:
> (l) «*I have 24 hamsters. I put them into some cages. Each cage contains 4. How many cages do I need?*» [37]

[31] It is advisable to make the pupils do their own translation before exchanging their texts with those of the classmates and making comparisons.

[32] (C) and (D) show that they grasp the existence of negative numbers. It is better to verify that it is not scarce control on the meaning of the sentence and to avoid and that the only concern for the pupil is not that of writing a sentence (often 'difficult') for the classmates.

[33] (C) translates in a correct but not significant way; it is better to make sure that the pupils can do a less 'literal' paraphrase.

[34] The translation from mathematical language into English implies more difficulties than its opposite. Even with younger pupils, it is better to favour reflection and verbalisation on the mathematical writings.

[35] The introduction of indefinite adjectives can lead to a very useful linguistic reflection, highlighting the equivalence of terms like 'a few', 'some', all of which to be referred to the meaning of 'a part of …'.

[36] It is advisable that the teacher often varies the context (geometrical, familiar, buy and sell, etc.).

*[37] The second situation is more difficult because in this moment of the school year the division is a very 'fresh' operation in the class.
(i) and (j) introduce a partition division, (h) a division of contenence; the missing number is difficult to handle and it blocks the general vision of the situation. The pupils try to avoid the obstacle using generic words like* some *and* I give them away.
*(k) introduces the 'partition into equivalent parts' but the connection to number 4 is obscure.
(l) solves the situation in a brilliant way, introducing the cages and proposing an actual partition division.*

It is interesting to compare the preceding and the following diary, which will inspire the successive variant of the Game of cross translations.

Diary 11 (primary school, 10 year olds, February)

Invent a problem that can be translated for Brioshi with the sentence: 26 – a = 11
Instead of 'simply' translating the sentence, the children turn it into a problem: «I have 26 candies and I eat 11; how many are left?» Some other attempts lead to a more faithful translation of the starting equation: (a) «I have 26 candies, I distribute *a few*, 11 are left» ✓ We ask what might substitute that 'a few'. (b) «A *number*». ✓ We ask to translate the sentence without turning it into a problem. Proposals: (c) «26 minus a number gives 11» (d) «I subtract a mysterious number to 26 and obtain 11» (e) «I subtract a number to 26 and obtain 11»

13. GAME OF CROSS TRANSLATIONS (second version)

The preceding version is centred on the translation from mathematical to natural language and vice versa; this version is centred on the passage from a 'real' to a formal context and vice versa.

Children work in pairs. Every child of the pair invents a problem (a datum must be missing) and writes on a piece of paper his/her mathematical representation. Then the children in the pair exchange their problems and they both represent the classmate's problem. In the end the two representations are compared. The most interesting cases are discussed collectively.

Examples (primary school, 10 year olds): **38**

Manuel: There are 50 cars, 20 of them are Fiat. How many of the cars are Peugeot?
Manuel's representation: 50 – g = 30
Blondino's representation: 50 – 20 = 30;   "test" 30 + 20 = 50

Davide: There are 20 turtles in 4 basins. How many groups are they divided into?
Davide's representation: 20: s = 4
Andrea's representation: 20: x = 4; 20 : 5 = 4

Andrea: There are 21 going to a town. They are divided in 3 squadrons. How many policemen are in each squadron?
Andrea's representation: 21: 3 = p
Davide's representation: 21: 3 = 7 **39**

Chiara: Mario collects 13 shells. Mario gives 2 to Marco. How many shells are left with Mario?
Chiara's representation: 13 – 2 = n   13 – 2 = 11
Debora's representation: 13 – 2 = 11

Debora: Mum buys 5 packets of candies, in all there are 15 candies. How many packets are contained in each packet?
Debora's representation: 5 × y = 15
Chiara's representation: 5 × m = 15; 5 × 3 = 15 **40**

Michael: At the supermarket there are 7 boxes of firecrackers. In all there are 28. How many firecrackers are there in each box?
Michael's representation: 7 × w = 28   e   w × 7 = 28
Nicole's representation: 28: 7 = 4

Nicole: I went to a pet shop and I bought 6 hamsters. Once I got home, I put 17 hamsters in a cage. How many hamsters did I buy in the other shop?
Nicole's representation: 6 + n = 17
Michael's representation: 6 + 11 = 17.

38 *In this context observations on the way in which texts are formulated are completely ignored (for example: in Manuel's problem it is given for granted that all the non Fiat cars are Peugeots, in Davide's problem the reference to 'groups' is ambiguous, in Andrea's it is not specified if all the squadrons have the same number of components, and so on. Of course these aspects – connected to the correctness and coherence of the text – are very important. They should take up the class's attention in a work of collective evaluation.*

39 *It is likely that teachers realise the presence of different attitudes in their class: when the pupils are* authors, *they tend to translate their text into arithmetic algebraic language introducing some letters in their writings; when the pupils themselves become* solvers *they often build* arithmetic *answers and they solve the problem instead of translating it for Brioshi (some pupils also verify their solutions).*
Probably the pupil-author feels free from preoccupations and/or stereotypes (he *invented the text, so he perceives it like a* real *problem), so he respects the instruction and 'translates'. On the other hand, in the second phase he's again victim of the stereotype that* 'the problem has to be *solved', and hence 'to look for the operation'.*
Activities that favour comprehension of the difference between the two attitudes and the reflection on them, so that the 'anxiety for operations' gives way to the translation from natural into algebraic language.

40 *Contrary to the attitude described in the preceding comment, Debora and Chiara prove an effective reflection on the instructions given. In fact, both as authors and as solvers, they use a correct algebraic language.*

Suitable age related activities	6	7	8	9	10	11	12	13	Comments

Fifth phase [41]

15. At the right moment we can propose more difficult problems, offering very interesting didactic hints.
Some examples are listed below. They are documented in the next pages with the Diaries (primary school, 10 year olds) that show some of the possible developments of this activity in the classes.

Alvaro and the cars (Diaries pages 18 - 20)

> Alvaro enjoys himself counting red and white cars. He counts 15 white cars, he doesn't remember how many red cars he counted, but he is sure that in all he counted 56 cars.
>
> Represent the situation in mathematical language so that it becomes a problem for Brioshi

Alice's library (Diaries pages 21 - 22)

> Alice is ordering his library. On a shelf he counts 16 Asterix books, 22 books of the 'Tales and Rhymes Einaudi' series and 11 of the 'Steamboat' series. She is very satisfies and says «Wow! I've got 49 books!»
>
> Represent the situation in mathematical language.

Caterina and the farm (Diaries pages 23 - 24)

> Together with her class, Caterina is visiting a farm. She counts 18 hens but she can't count the doves because they go in and out of the aviary. The farmer tells them that in all he has 43 birds.
>
> Represent the situation in mathematical language so that it becomes a problem for Brioshi

Note 2

In reading the **diaries** in the next pages we must keep in mind that everybody is referring to particular situations of the class and also the comments are made 'on the spur of the moment'; the aim of the diary is that of furnishing significant materials to the teachers (experimenters) with different objectives:

(i) periodically inform about what happens in the class during the presence of the researcher with the teacher;
(ii) give indications about the method (discussion, comparison of protocols in class, subtle analysis that allows deeper readings and so on);
(iii) suggest hints for reflection and hypothesis for activities in the column of comments;
(iv) control on the hypotheses of work programmed at the beginning of the school year.

[41] As the horizon of activities widens and pupils become aware of these subjects, it is important to take advantage from their natural curiosity and to illustrate codes, conventions, symbols, etc.
Good occasions are offered by protocols mixing symbolic, natural and iconic (or invented by teachers) language in a confused or inappropriate way. In other words, all those representations that Brioshi couldn't understand.
Therefore, we can introduce (in terms suitable for the age) the concept of "universal language" (or as somebody named it, a 'mathematical esperanto').
We can explain that students, often not experts of mathematical language, use it with fantasy, also in a 'beautiful', but not comprehensible way (often pupils think they understand or guess the intentions of the authors of protocols, because they know the 'mathematical dialects' used in their class; but Brioshi, alien to all these dialects, wouldn't know how to interpret them).
Another possible remark might be that often the authors themselves were not very clear in the use of language, and later on they cannot remember what they meant with a certain representation.
Referring to codes which are familiar to the class is another useful hint. The Morse code, for example, was invented to save people from shipwrecks and their messages had to be clearly understandable by anybody, whatever language they spoke. If the operator transmitted non-recognisable signals, the shipwrecked would not have been saved, because nobody would understand the message.
The aim is that of making the pupils understand the historical reasons why mathematical languages, as well as the other languages, have undergone some changes during the course of time. Mathematical languages are much poorer than natural language, but they are very powerful in their conciseness if they are used correctly.

ArAl Project	U1. Brioshi and the approach to algebraic code

Ed	*1650a*	*Problems for Brioshi (diary)*				4				

Borgo Piave school, 10 year olds, November 23rd 1999, first meeting	*Comments*

✓ We introduce Brioshi; then we propose Alvaro's problem.

> While driving, Alvaro enjoys himself counting red and white cars. He counts 15 white cars, he doesn't remember how many red cars he counted, but he is sure that in all he counted 56 cars.
> (a) Represent the situation in mathematical language so that it becomes a problem for Brioshi

At first pupils translate the problem in purely iconic language, with coloured, detailed and bizarre representations.
✓ We ask for a translation into a language that can be sent via e-mail.
After a series of doubts, the pupils give the following answers:
(a) 56 - 15 =
(b) 15 + …. = 56
(c) 56 - 15 = 41 15 + 41 = 56 41 + 15 = 56 56 - 41 = 15
(d) 15 + ? = 56
(e) 15 + □ = 56
(f) 56 - 15 = ?
(g) 56 = 15 + ….[42]
(h) 56 - 15 = …..
(i) 15 + = 56
The answers are written on the blackboard.
✓ «In your opinion, what is the most suitable to send to Brioshi?»
(e) is immediately rejected «… because the square creates confusion» and also (i) («How can we send a message like this? The computer might have some problems» - *they refer to the blank space, ndt)).*
This discussion follows:
«In my opinion (c) is fine, because it's more complete»
«Oh no, (c) has no problems, there are only numbers with the result»
«(f) or (g): with the question mark or the dots, it is clear that we don't know the number, and so it is a problem»
«(g) is number 56 divided: and here come the operations I have to make to get 56, in (f) there is a subtraction. (f) is an operation, (g) is a problem»
«(f) can be the question of a problem, (g) is only the test»
«(f) is clearer, (c) is already completed»
« (f) and (h) are simpler operations than (g), but they are also the same»
«In (g) you have to find»

Almost all the pupils intervened, anyway everybody has listened.
We decide to classify the pupils' answers on the basis of the intentions of the solver:

1. The problem is solved: (c)
2. The problem has still to be solved: (b), (d), (e), (i)
3. Operations are privileged: (f), (a), (h).

We decide to send a 'problem'.
A pupil gives a very interesting answer «The solution (g) is more *problematic* than (b)» [43]

(Continues in the following page)

[42] *(b) and (g)(which are equivalent and symmetric) are felt by the pupils as different. This might happen because – as will result clearly from the following discussion – the equal sign is seen as a* <u>*directional operator*</u> *and not as an indicator of relations of equivalence.*

[43] *We suppose that (g) is seen as 'more problematic' because the result of the operation is on the left of the equal sign.*

Ed	*1650b*	*Writing in column and writing in line (diary)*				*4*				

Borgo Piave school, 10 year olds, 24 pupils, November 23rd 1999, meet. ***1***	*Comments*

→

There is a lot of uncertainty on what is the formulation to send, among those of the second group. Another two formulations emerge:

(j) 15 + x = 56
(k) 56 = 15 + ?

✓ We ask to explain why an ‘x’ is added.
No answer is given, but the other pupils disagree, because they say that the ‘x’ creates confusion with the operation of multiplication.
At this stage the class decides to send formulation (k) **44**.
✓ «In your opinion, what will Brioshi answer?»
Many proposals emerge from the discussion:

(a) 56 = 15 + ? 56 - 15 = 41 (in column) 56 = 15 + 41

$$\begin{array}{r} 56\ - \\ \underline{15} \\ 41 \end{array}$$

(b) 56 = 15 + 41

(c) 56 = 15 + ? 41 = 56 – 15

$$\begin{array}{r} 56\ - \\ \underline{15} \\ 41 \end{array}$$

(d) 56 - 15 = 41
(e) 41 + 15 = 56

✓ We ask to formulate the answer in natural language, and to translate then this message into mathematical language.
The pupils’ answer comes without apparent difficulty:

“Alvaro counted 41 red cars”

and the translation:

? = 41

After giving this answer, the pupils don’t even consider it, as if it had nothing to do with the discussion, or worse as if it was a hindrance. They go back to the discussion on what formulation to choose.
The meeting ends without a choice of the most suitable formulation.

44 The scheme ‘text - solution - answer’ fits the instruction very well.

ArAl Project	U1. Brioshi and the approach to algebraic code

Ed	*1651*	*Writing in column and writing in line (diary)*				4				

Borgo Piave school, 10 year olds, 24 pupils, November 23rd 1999, meet. ***2***	*Comments*

Alice's problem is presented:

> Alice ordering her library. On a shelf she counts 16 Asterix books, 22 books of the 'Tales and Rhymes Einaudi' series and 11 of the 'Steamboat' series. She is very satisfied and says «Wow! I've got 49 books!»
> Represent the situation in mathematical language for Brioshi.

Here are the answers given by the children:

(a) 22 + 11 + 16 = 49
(b) 16 + 22 + 11 = 49
(c) 49 - 11 - 22 - 16 = 0
(d) 16 + 11 + 22 = 49
(e) 16 + 11 = 27   27 + 22 = 49 **45**
(f) 22 + 11 + 8 + 8 = 49
(g) (10 + 6) + (20 + 2) + (10 + 1) = 49
(h) 11 + 11 + 11 + 8 + 8
(i) 49 = 11 + 16 + 22
(j) 49 - 11 - 22 = 16
(k) 22 + 16 + 11 - 49 = 0
(l) ? + 11 + 22 = 49
(m) 16 ? + 11 = 49 **46**

Furthermore, there are two operations in column:

(n)	(o)
16 +	16 +
22 +	11 +
<u>11</u> =	<u>22</u> =
49	49

The representation are several, the problem is to choose the most suitable one to represent Alice's story.
Writings (l) and (m) are eliminated immediately because «In the text there are no questions, these sentences represent some problems».
Also (e) is quickly eliminated because in the text of the problem there are no intermediate solutions.
Also (f), (g) and (h) are quickly eliminated because «Numbers are decomposed, and they don't tell the story. Moreover, (h) is lacking a result.
The following proposals to be eliminated are those represented by writings (c), (j), (k), because «In the problem Alice doesn't take anything away, and so these mathematical sentences don't tell the story».
Then we decide to maintain only the writings in additive form, no matter whether they are in column or in line. In this regard, (b) and (n) are coupled, just like (d) and (o), because they represent the same operation in column and then in line **47**.
The remaining sentences are: (a), (b), (d), (i), (n) e (o).
It is evident that (i) is **<u>structurally</u>** different from the others.
The pupils don't want to eliminate it, but they are very doubtful. In fact, having the result of the operation on the left of the equal sign is unusual, because generally the result is on the right. At the beginning Alice tells us how many books she has for each series, and then she tells us the total number. Therefore, we conclude that the remaining writings in line, that are (a), (b), (d) and (i) are mathematically equivalent, but (b) is the most faithful one to the order of the story.

45 We might take advantage of a situation like this (e) to face the concept of binary operation, because whatever is the number of addends, they are anyway coupled.

46 Notice in (m) the absence of an operation sign between the number and the unknown.

47 This might be an occasion to speak about the operations, which hold the commutative law.

Ed	*1605a*	***Writing in column and writing in line (diary)***				4				

S.Giustina school,10 year olds, 24 pupils, February 8th 2000, meeting ***5***

We propose this problem:

> Alice is ordering her library. On a shelf she counts 16 Asterix books, 22 books of the 'Tales and Rhymes Einaudi' series and 11 of the 'Steamboat' series. She is very satisfied and says «Wow! I've got 49 books!»
> Represent the situation in mathematical language for Brioshi.

We receive the following protocols, extremely various:

(a) 16 $\boxed{\text{Asterix}}$ 22 $\boxed{\text{Tales}}$ 11 $\boxed{\text{Steamboat}}$

(b) 16 + (5 pupils)
22 +
11
49

(c) [48] 1($\boxed{\text{Asterix}}$ + 22 $\boxed{\text{Tales}}$ + 1 $\boxed{\text{Steamboat}}$ = 49

(d) 16 + 16 + 22 + 11 = $\boxed{}$
22 +
11
49

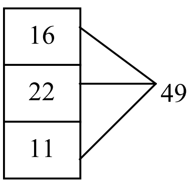
(e) 16 + 16 + 22 + 11 = $\boxed{49}$
22 +
11
49

(f) 16 + 22 + 11 = 49 *(3 pupils)*

(g) 16 + 22 + 11 = 49 16 +
22 +
11
49

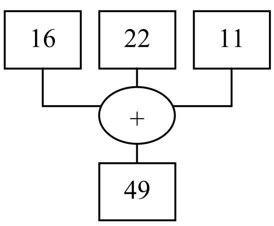
(h) 22 +
16 +
11
49

(i) [49]



(j) [50] 22 + 16 + 11 = 49

22 +
11 +
16
49



49 – 22 – 16 - 11

(Continues in the following page)

Comments

[48] We cannot understand the reasons for the representations (a) and (c) the pupil (a) probably copied from (c); they are pupils' 'inventions', which the authors themselves cannot explain afterwards.

[49] Also the girl who wrote (i) seems to have 'invented' her representation.

[50] The teacher notices that the author of (j) usually produces redundant texts, representing all the strategies without attributing any strategy within them.

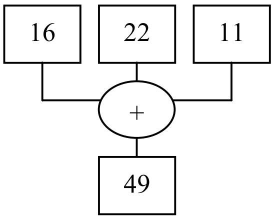
ArAl Project	U1. Brioshi and the approach to algebraic code

Ed	*1605b*	*Writing in column and writing in line (diary)*				4				

S.Giustina school, 10 year olds, 24 pupils, February 8th 2000, meeting ***5***

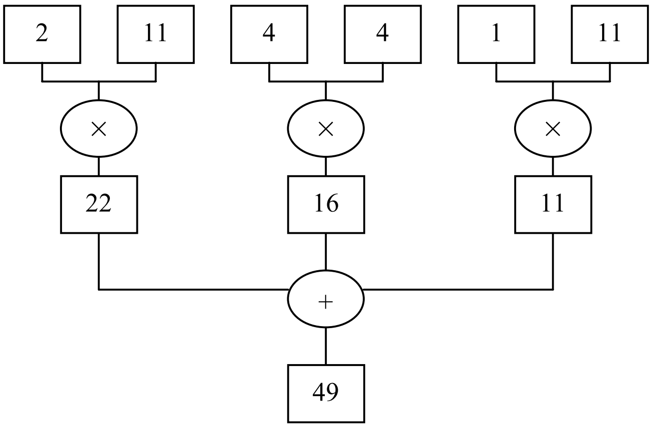
→

(k) 16 +
22 +
<u>11</u>
49



Test: 11 +
16 +
<u>22</u>
49

(l)



22 + 16 + 11 = 49

22 +
16 +
<u>11</u>
49

(m) 16 + 22 + 11 = 49

22 + 11 + 16 = 49
11 + 16 + 22 = 49
16 + 11 + 22 = 49

(n) 22 + 11 + 16 = 49
16
<u>22</u>
49

During the discussion the following representations are gradually excluded:
- (a) and (c) because they are not written in mathematical language. Moreover, in (a) « the operator is missing».
- (d) because it does not correspond to the instruction, as «there are no mysterious numbers1
- (e) the square around 49 is not necessary.
- A discussion begins on (l); teachers ask every pupil to explain his scheme, but it is clear that the initial data have been chosen arbitrarily.

The meeting is over.

Note

The activity in presence of both teachers continues in the following meeting (see **Ed 1606**, pages 23 - 24).

Comments

Conclusive observations

Three types of representations appear, and they are often present in the same protocol:
- *in column,*
- *in line,*
- *iconic (often graphs)*

Generally the pupils don't seem to choose a representation functional *to a purpose; when given an unusual instruction, they "perform" the representations they know. They even use the stereotype of the 'test' (k) (and maybe (j)) or of an inadequate application of the commutative property (m).*
When the known instruments are poor, they recur to invention: (a), (c) and (i).
In any case the principle of redundancy prevails, and in some cases it is even exaggerated (j).
In some cases (l) there is total loss of control on the meaning; the pupil feels that 'something is missing' and invents non-existent data that he/she cannot justify later on.
Only three pupils (f) respect the instruction and represent the situation with a writing in line.
Five pupils (b) present 'clean' protocols, but they appear to be more linked to the sequence 'calculation –result - solution' than to the concept of 'translation – representation' of the relations.

These partial results are very similar to those obtained in the fourth year of Borgo Piave School (see. ***Ed 1651****, pages 18 – 20). The pupils' disorientation is evident and so is the resort (often uncontrolled) to the most commonly used representation for the solution of problems.*

Ed *1606a* *Writing in column and writing in line (diary)* 4

*S.Giustina school, 10 year olds, 24 pupils, February 22nd, 2000, meeting **6***	*Comments*

We continue the collective correction of the protocols of the preceding file regarding Alice's problem (**Ed 1605**), pages 21 – 22).

(a), (c), (d), (e) have already been excluded; now they are eliminated:
(g) because they repeat writings of other pupils;
(h) because they are similar to (b) with inverted data;
(i) their meaning is not clear;
(l) because the initial data put in the file are arbitrary;
(m) because it is useless to rewrite the same expression three times consecutively using the commutative property;
(n) because they don't make much sense (the look like 'crosswords').

About (k) and (j) a long discussion begins because according to the pupils the presence of more representations «allows to understand better». We discuss on this statement but we don't obtain convincing answers. In the end, the class agrees on the fact that clear writings are understood even if there are no other writings to compare them with, but it is evident that the agreement is made basically to please the teacher. Only one pupil seems to be convinced that many writings in this case are superfluous [51].

Only two representations remain on the blackboard: (b) and (f):

(b) 16 +
22 +
<u>11</u>
49

(f) 16 + 22 + 11 = 49

✓ We ask the pupils if they consider these writings as less clear because, as someone said 'they are alone'.

Alvise (author of m) explains that he wanted to «represent the situation in other ways»; anyway, he admits that (f) is clear.

Manuel repeats that it is superfluous to say the same thing in different ways.

✓ We ask which of the two is the best to send to Brioshi.

At first the pupils seem to be very doubtful on what to do.

✓We ask to raise hands in favour of one or the other.

Initially, only three or four pupils vote in favour of (b). Then others add up and in the end we count 16 raised hands.

Seven pupils vote for (f).

During the discussion (f) is defined as «the most beautiful», «the most difficult», «it is more a problem for Brioshi» [52].

Other interventions help to arrange the 'frame' within which the pupils put the representation 'in line' and that 'in column' when they solve a problem. In their opinion, the protocol is the following:

1) The operation is written in line up to the <u>equal sign</u>;
2) We move to the other side of the page;
3) We write numbers in column and calculate the result;
4) We go back to the operation in line and add the result, on the right side of the equal [53].

(Continues in the following page)

[51] *As underlined in **Ed 1605b**, for many pupils the choice of a representation is not a really aware one (basically, we talk of a choice, which is functional to the supposed expectations of the teacher of 'exhibiting' a representation).*
Teachers ought to think over this difficulty: the choice of a representation must be functional to what? *Probably, if hesitating, pupils recur to all the representations that they know somehow (in line, in column, graphical) more or less clear, often drawn from confused memories.*
In other words, pupils resort to known techniques, *leaving the teacher to choose the most* suitable.
We suppose that also the observation made by someone «it helps to understand better» does not express a real persuasion. In fact, in a few minutes the class reverses completely the conclusions drawn before (and even this last decision does not express a real awareness).

[52] *(f) is described as 'more difficult' because the numbers are not in column; 'more beautiful' meaning maybe more mature, 'more adult'.*

[53] *The two representations appear to be technical facts, poor in meaning. A reflection should be made with the teachers on the writing in line as a 'embryo of expression' and see how this point of view can positively influence also that of the pupils.*

ArAl Project	U1. Brioshi and the approach to algebraic code

Ed	***1606b***	***Writing in column and writing in line (diary)***				**4**				

S.Giustina school, 10 year olds, 24 pupils, February 22nd 2000, meeting ***6***	*Comments*

→

Together with the teacher we observe that this protocol reflects a *social practise* consolidated within the class that conditions the attitude of the whole class.
As an appendix to the discussion, Manuel observes that it is not obvious that the operation in line is more difficult than that in column.
We propose a new problem, but we give a new didactic agreement:

- every pupil will write a single representation; in case of doubt, he will have to choose the most suitable one;
- no property has to be applied;
- no graphical schemes have to be used.

We propose Caterina's problem:

> Together with her class, Caterina is visiting a farm. She counts 18, but she cannot count the doves because they go in and out of the aviary. The farmer tells them that in all he has 43 birds.
> Translate the situation in mathematical language for Brioshi.

Distribution of the answers:

(a) 11pupils solve the calculation and find the missing number.

(a_1) 43 – (4 pupils)
<u>18</u>
25

(a_2) 43 – 18 = 25 (5 pupils)

(a_3) 18 + 25 = 43 (2 pupils)

(b) 7 pupils translate the problem for Brioshi:

(b_1) 18 + $\square$ = 43 (3 pupils)

(b_2) 18 + ___ = 43 (3 pupils)

(b_3) 43 -
\> a [54]
18 =

[54] A striking observation is that only one answer (b_3) contains a letter, moreover inserted in very 'unclean' writing.
Blank spaces and squares still appear. The fourth meeting with the class- during which the pupils had made an abundant use of letters – seems to have left no significant traces.

Hypothesis*: The use of the letter as a mysterious number has not to be occasional and also it has to be introduced very early as time goes by pupils adopt strong habits and beliefs, very difficult to change.*