Unit 2

Numbers grids

ArAl Project	U2. Numbers grids

1. The Unit

The work on grids has been inspired by some activities proposed in the English *The National Numeracy Strategy* published by the *Department for Education and Employment.* It started in the school year 1998/99, and has become more and more important in the ArAl project as an approach to equations and training material for **pre-algebraic** thought, by autonomously adding original hints and suggestions.

2. Didactic aspects

The unit starts from the building of a square grid, numbered from 0 to 99 and, through more and more sophisticated activities leads the pupil to explore:

- the **polynomial representation** of the number
- the operations and their properties
- the addition and multiplication operators and their **composition**
- the mental calculation strategies
- the **regularity**
- the search for conditions of solvability for problems.

The activity is also developed in the algebraic field, through problems, which are similar to the preceding arithmetic ones, although proposed in a parametric form.

3. General aspects

- The unit can be developed in every class of primary school, according to the modalities illustrated in paragraph **6**.
- Starting from the initial phases, the Unit is divided into problems; the class – divided in groups or through individual activities –faces more and more complex situations and tries to solve them. The **verbalization** and the **collective** comparison of the strategies adopted allow to spread and reinforce the results of *discoveries*.
- The solution of problems always leads to collective discussions, which become very important because they force everybody to think about his mental processes, to express his thoughts and strategies, to listen to the others, thus exalting not only the cognitive aspects but also the metacognitive and metalinguistic ones.
- The problems are exciting and are often presented as games, as intellectual challenges.
- It is very simple to find new problems or with a structure similar to that of the problems already submitted to the class.
- The unit contains problems with one solution, with more than one solution, with infinite solutions, impossible problems, thus helping to undermine the stereotype of problems with one solution.

4. Terminology and symbols

Phase	Sequence of situations of growing difficulty referred to the same subject.
Situation	Problem around which individual, group or class activities are developed.
Expansion	Hypothesis of work on a possible expansion of the activity towards an algebraic direction. Its realisation depends on the environmental conditions and on the teacher's objectives.
Supplementary activity	Enlargement towards subjects related to those developed in the preceding **Situations**.
Note	Methodological or operational suggestions for the teacher.
	In the square a problematic situation is proposed. The text is purely indicative; it can also be presented as it is, but generally its formulation represents the outcome of a social mediation between the teacher and the class
<u>Representation</u>	An underlined word in boldface type highlights a link to a subject illustrated in the Glossary.

Square containing the outline of a typical **<u>discussion</u>**; the following symbols may appear:

√	Intervention of the teacher
	Intervention of a pupil
	Summary of several interventions
	Summary of a collective discussion (a principle, a rule, a conclusion, an observation, …)

5. Phases, situations and subjects

All the phases are abundantly supported by significant extracts from the **<u>Diaries</u>** of activities in the classes.

PHASES	SITUATIONS	SUBJECTS
First	**1 – 6**	Building of the grid 0 – 99 and analysis of the regularities appearing: necessary activity in every class to start the work.
Second	**7 – 10**	Strategies for mental calculation starting with the grid.
Third	**11 – 13**	Activity with an uncomplete grid: application of regularity, exercises for mental calculation, similarity as an equivalence **<u>relation</u>**, rules to solve expressions.
Fourth	**14**	Activities with pieces of grid: use of regularities identified in the preceding situations.
Fifth	**15 – 17**	The island game: problems-games, even with cases of problems without a solution; impossible problems and study of the conditions to solve them.
Sixth	**18 – 20**	Start of generalisation. Positional notation, polynomial representation of a number.
Seventh	**21 – 25**	Approach to **<u>letters</u>** as parameters.
Eighth	**26 - 32**	Problems of application. Extension to parametrical problems and to grids of n × n dimension. Activities with fragments of grids to be analysed and completed.

6. Distribution of the situations in relation to the pupils' age

The distribution represents an indicative proposal based on the experience in the classes taking part in the project. Every situation can be restricted or widened according to the methodological choices or to the opportunities of the teacher.

<table>
<tr><th colspan="3"></th><th colspan="29">PHASES AND SITUATIONS</th></tr>
<tr><th colspan="3"></th><th colspan="6">I</th><th colspan="4">II</th><th colspan="3">III</th><th>IV</th><th colspan="3">V</th><th colspan="3">VI</th><th colspan="5">VII</th><th colspan="4">VIII</th></tr>
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ArAl Project	U2. Numbers grids

Suitable age related activities	6	7	8	9	10	11	12	13	Comments

First phase

1. The activity is developed on the basis of a grid numbered from 0 to 99 [1]

0	1	2	3	4	5	6	7	8	9
10	11	12	13	14	15	16	17	18	19
20	21	22	23	24	25	26	27	28	29
30	31	32	33	34	35	36	37	38	39
40	41	42	43	44	45	46	47	48	49
50	51	52	53	54	55	56	57	58	59
60	61	62	63	64	65	66	67	68	69
70	71	72	73	74	75	76	77	78	79
80	81	82	83	84	85	86	87	88	89
90	91	92	93	94	95	96	97	98	99

2. The initial questions invite to the exploration of the grid. For example:

- Which are the smallest and the biggest number of the grid?
- How many lines are there?
- How many columns are there?

3. The **Game of Rules**: the pupils must discover and make explicit the relation between the numbers of any

- line (+1, -1)
- column (+ 10, - 10)
- diagonals N.E. – S.W. (+ 9, - 9)
- diagonals N.W. – S.E.[2] (+ 11, - 11).

The exploration clarifies that each direction has two sides.
It is useful to make the pupils <u>**represent**</u> the situations with the help of graphs; this is done in order to reflect on the operations and their inverse. For example:

22 $\underset{-1}{\overset{+1}{\rightleftarrows}}$ 23 $\underset{-1}{\overset{+1}{\rightleftarrows}}$ 24 $\underset{-1}{\overset{+1}{\rightleftarrows}}$ … 3rd line

41 $\underset{-10}{\overset{+10}{\rightleftarrows}}$ 51 $\underset{-10}{\overset{+10}{\rightleftarrows}}$ 61 $\underset{-10}{\overset{+10}{\rightleftarrows}}$ … 2nd column

38 $\underset{-9}{\overset{+9}{\rightleftarrows}}$ 47 $\underset{-9}{\overset{+9}{\rightleftarrows}}$ 56 $\underset{-9}{\overset{+9}{\rightleftarrows}}$ … 3rd diagonal NE-SW [3]

54 $\underset{-11}{\overset{+11}{\rightleftarrows}}$ 65 $\underset{-11}{\overset{+11}{\rightleftarrows}}$ 76 $\underset{-11}{\overset{+11}{\rightleftarrows}}$ … 4th diagonal NW-SE [4]

[1] It is advisable to prepare a poster and little individual grids to stick on the copybooks every time.
A grid 1 - 100 might also be used; we prefer the 0 - 99 one, because it accustoms the pupils to the presence of nought.

[2] During the second year of primary school, it might be necessary to substitute the terminology connected to the cardinal points with less specialised words like ‘on the right top’ or ‘at the left bottom’.

[3] The verbalization, the comparison and the analysis of the definitions given by the pupils are very important. Two examples from the third year; in the first one the class is at the beginning of the ArAl activity, in the second the class has already worked with the pyramids (see U3 Numbers Pyramids).
(a)«To pass from 38 on to 56 I counted and then I divided 18 by 2»
*(b) «For <u>**Brioshi**</u> I would write 38 + n = 56 and so n = 18» (very interesting proposal even if incorrect).*

[4] In describing the ‘rule’ (also defined as ‘step’) younger pupils often use sentences like «The units go on *one by one» or «The units* grow *one by one».*
It is a typical 'concrete' descriptive *attitude; it presupposes the use of verbs giving an idea of* action*: ‘go on’, ‘grow’, ‘increase’. Similarly, Italian pupils in the first year of intermediate school use for geometry terms like: «the segment* starts *from and* arrives *at».*
Teachers can take advantage of these occasions to strengthen the linguistic competence by introducing or reinforcing the concepts of preceding and following.

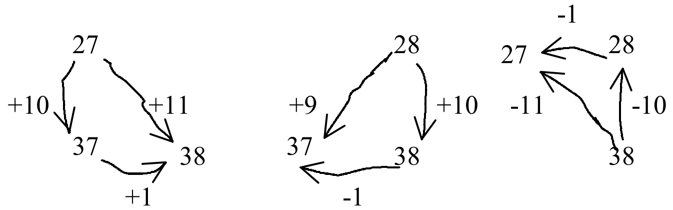
Suitable age related activities	*6*	***7***	***8***	***9***	***10***	***11***	***12***	***13***	*Comments*

4. Starting from a fragment of the grid, we can explore the operating rules which allow the shifts from a square to another [5].
For example:

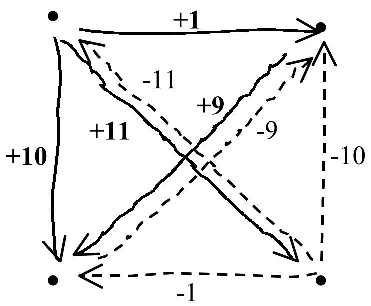
(a) from 27 to 38
(b) from 28 to 37
(c) from 38 to 27

(a) da 27 a 38
(b) da 28 a 37
(c) da 38 a 27

27	28
37	38



5. We build a graph that illustrates the possible steps within the grid. We can lead the class to understand that this is valid in any area of the grid. In this way is not necessary to write the numbers in the nodes of the graph anymore, because it assumes a general validity, at least in the 0-99 grid, which we are operating with [6].



[5] *It might happen that not all the pupils grasp immediately the analogies between the two representations (a double entry chart and a graph).*
It is suggested that the teacher makes sure whether this awareness is lacking and if necessary he/she intervenes with the suitable clarifications.

[6] *In the **Sixth phase** square grids will be introduced, with a variable number of lines and columns.*

ArAl Project	U2. Numbers grids

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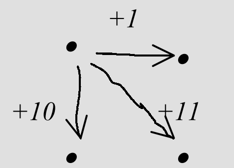
6. The exploration of the grid and the contemporary **collective** **discussion** lead to a continuous building of definitions which illustrate the discovery of regularity. The comparison between the definitions helps to identify the best ones as for correctness and clearness.
Some examples (third year, primary school):

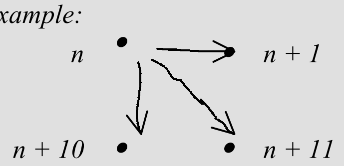
- «Along a line, we move +1 on the right, -1 on the left»
- «Towards the right the order is increasing, towards the left it is decreasing»
- «Along every column the rule is + 10 is down and – 10 is up»
- «Along any oblique arrow NW - SE the step is + 11 or – 11»
- «Along any oblique arrow NE – SW the step is + 9 or – 9»

Expansion 1

● *Hypothesis of work with pupils of the fifth year (primary school, 10 year olds) – first year (intermediate school, 11 year olds): guide to the concept of sum of vectors*



● *Hypothesis of work with pupils of the fifth year (primary school, 10 year olds) – first year (intermediate school, 11 year olds): get to the algebraic description of the nodes of the graph in function of one of them, for example:*



● *An observation can be made, from the fourth year (9 year olds) on, that the operators are composed following the laws of arithmetic. For example:*

$$\xrightarrow{+10} + \xrightarrow{+1} = \xrightarrow{+11}$$

With the help of generalisations, we discover that:

$$\xrightarrow{+a} + \xrightarrow{+b} = \xrightarrow{+a+b}$$

$$\xrightarrow{+a} + \xrightarrow{-b} = \xrightarrow{+a-b}$$

but also that - which is very important:

$$\xrightarrow{+a} + \xrightarrow{\times b} \neq \xrightarrow{\times b} + \xrightarrow{+a}$$

that is, the composition of operators does not hold the commutative property

Suitable age related activities	6	7	8	9	10	11	12	13	Comments

Second phase

7. Individuation of the 'step' in paths leading 'directly' from a starting number to a final one. Some examples [7]:

0	1	2	3	4	5	6	7	8	9
10	11	12	13	14	15	16	17	18	19
20	21	22	23	24	25	26	27	28	29
30	31	32	33	34	35	36	37	38	39
40	41	42	43	44	45	46	47	48	49
50	51	52	53	54	55	56	57	58	59
60	61	62	63	64	65	66	67	68	69
70	71	72	73	74	75	76	77	78	79
80	81	82	83	84	85	86	87	88	89
90	91	92	93	94	95	96	97	98	99

[7] *From 6 to 33: with step +9*
From 36 to 69: with step +11
From 56 to 51: with step –1
From 94 to 50: with step –11

8. The grid can be used for activities of reflection and reinforcement of mental calculation, because it helps to visualise different paths – and hence different strategies - to 'pass' from a number to another. For example, in order to answer the question «How much shall I add to 23 to obtain 79?» at least three different paths can be identified (of course there are many more, but they are less economic) [8, 9].

0	1	2	3	4	5	6	7	8	9
10	11	12	13	14	15	16	17	18	19
20	21	22	23	24	25	26	27	28	29
30	31	32	33	34	35	36	37	38	39
40	41	42	43	44	45	46	47	48	49
50	51	52	53	54	55	56	57	58	59
60	61	62	63	64	65	66	67	68	69
70	71	72	73	74	75	76	77	78	79
80	81	82	83	84	85	86	87	88	89
90	91	92	93	94	95	96	97	98	99

[8] *Once again we underline the importance of verbalization and of comparison between the expressions used by the pupils. For example, the path (thick line) of situation 8 is described in different ways by pupils of the third year (primary school):*

- *«I add to 23 five times 11 and then I add 1»*
- *«I add 11 five times and then I add1»*
- *«I sum 23 and 55 and then I add 1».*

The discussion can highlight the superior ***transparency****, as far as the* ***process*** *is concerned, of the first two strategies if compared to the third one.*

[9] *It might happen, especially with younger pupils, that the starting square is included in the calculation of the steps, and hence that law is applied once too often. For example in the blue path someone states that regularity +11 is applied* six times.

Suitable age related activities	*6*	*7*	***8***	***9***	***10***	***11***	*12*	*13*	*Comments*

9. Collective activity: we pass on to write down calculation which have been performed mentally so far; the complexity of this passage varies according to the pupils' age. For example, the paths of **situation 8** can be expressed in these forms:

thin line: $23 + 50 + 6$ or $23 + 10 \times 5 + 1 \times 6$
dotted line: $23 + 6 + 50$ or $23 + 1 \times 6 + 10 \times 5$
thick line: $23 + 55 + 1$ or $23 + 11 \times 5 + 1$ [10]

The second representations are more transparent in reference to the process.

[10] We can take advantage of these situations to face subjects like the relationship between the use of parenthesis and the priority of operations in arithmetic expressions.

10. Individual activity on the writing of mental calculations followed by a comparison between the various **<u>writings</u>** and by a collective discussion [11]. Some numbers on the grid are chosen and the different ways to pass from one to the other are represented; also indirect paths are considered.

[11] We can take advantage of this situation for some reflections on the organisation of mental calculation and on the effectiveness of 'clever' strategies to perform them more easily (in the maths field this is called economy principle).

0	1	2	3	4	5	6	7	8	9
10	11	12	13	14	15	16	17	18	19
20	21	22	23	24	25	26	27	28	29
30	31	32	33	34	35	36	37	38	39
40	41	42	43	44	45	46	47	48	49
50	51	52	53	54	55	56	57	58	59
60	61	62	63	64	65	66	67	68	69
70	71	72	73	74	75	76	77	78	79
80	81	82	83	84	85	86	87	88	89
90	91	92	93	94	95	96	97	98	99

Some examples of possible problematic situations

> Represent the operations, which allow to pass from 63 to 85 and viceversa [12].

[12] In order to clarify the meaning of the verb 'represent' in the instruction, we can recur to Brioshi, explaining that he has to reproduce the paths on his grid. In order to help him, we must explain all the calculations to be performed in mathematical language (see Unit 1 'Brioshi and the approach to the algebraic code').

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Paths identified by third year pupils (primary school, 9 year olds):

1. dotted line

63 + 10 + 10 + 1 + 1 = 85	85 – 10 – 10 – 1 – 1 = 63
63 + 10 × 2 + 1 × 2 = 85	85 - 10 × 2 - 1 × 2 = 63
63 + 20 + 2 = 85	85 – 20 – 2 = 63

2. thick line

63 + 1 + 1 + 10 + 10 = 85	85 – 1 – 1 – 10 – 10 = 63
63 + 1 × 2 + 10 × 2 = 85	85 - 1 × 2 - 10 × 2 = 63 [13]
63 + 2 + 20 = 85	85 – 2 - 20

3. thin line

63 + 11 + 11 = 85	85 – 11 – 11 = 63
63 + 11 × 2 = 85	85 - 11 × 2 = 63
63 + 22 = 85	85 – 22 = 63

> Represent the operations, which allow to pass from 16 to 63.
> Parenthesis can be introduced.

Pupils represent a long dotted line in different ways:

(a) 16 + 10 + 10 + 10 + 10 + 10 – 1 – 1 –1 = 63
(b) 16 + 10 × 5 – 1 – 1 –1 [14] = 63
(c) 16 + 10 × 5 – 1× 3 = 63
(d) 16 + 50 – 3 = 63

[13] *We can speak of priorities in the operations and of the use of parenthesis.*

[14] *In the fifth years it is important to make the pupils realise the equivalence between the writings:*

- 1 -1 -1 = - (1 + 1 + 1)

which allows to represent path (b) in another different way with the introduction of parenthesis:

16 + 10 × 5 – (1 + 1 + 1) = 63.

Expansion 2

Through the activities referred to in note 4, whether the environmental conditions allow it, the teacher can propose a generalising approach:

$$-a - b = -(a + b)$$

Suitable age related activities	*6*	*7*	***8***	***9***	***10***	***11***	*12*	*13*	*Comments*

Third phase

11. The activity can continue with an incomplete grid, like the one represented in the drawing. The teacher asks to insert a number in a blank square and the pupils must place arguing their strategy.

		2							9
10									
	21								
				34			37		
					55				
			63						69
						76			
	81								
							97		

√ [15] Explain how to place number 49.
- «I put it 20 units above number 69.»
- «I counted 2 units after 37 and then I went down 10.»

√ Explain how to place number 25.
- «I counted 21 and then 22, 23, 24 and 25»
- «I 'did' 34 minus 10 and then plus 1»
- «I start from 55 and go up, subtracting 3 times 10»

12. Tests and reinforcement activities with an *empty* grid (except for the squares containing 0 and 99). Pupils must place some given numbers correctly and explain the strategies they used [16].

0									
									99

[15] *The examples quoted are referred to third years (primary school, 8 year olds).*

[16] *It is advisable to make the pupils represent numbers also in a* ***non canonical*** *form, in order to mirror the mental calculations performed to identify the corresponding square.*
For example: when asked to place number 36, a third-year pupil (8 year old) explained the path followed in this way: «I go down 3 tens under 0 and then I move right of 6 units». The related representation was:

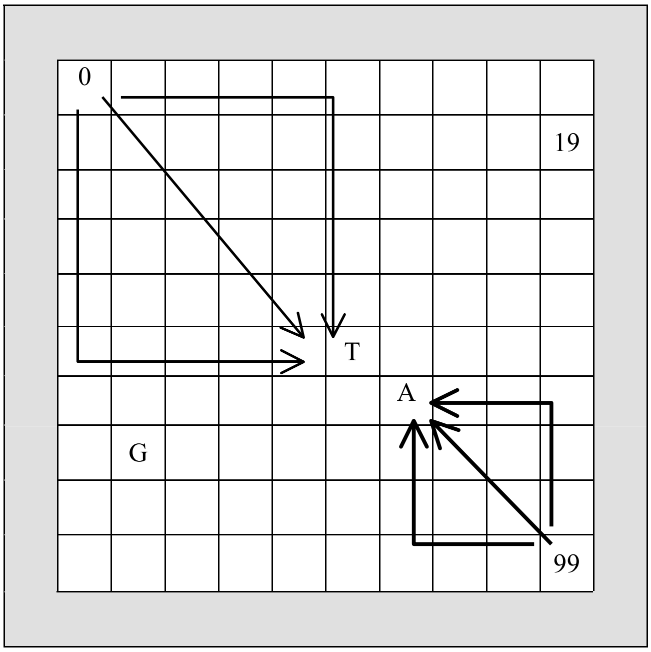
$$10 \times 3 + 1 \times 6 = 30 + 6.$$

Suitable age related activities	6	7	**8**	**9**	**10**	**11**	12	13	Comments

13. Game: treasure hunt

The almost completely empty grid represents an island surrounded by a sea in which some pirates hid a treasure (in example T).

The aim is that of making the pupils produce and compare different representations of a same path in order to find their equivalence; they must be made using the known 'steps' (+ 1 e – 1, + 10 and – 10, etc.) [17].



The game can be developed in various ways; we suggest three:

A) Children identify themselves with the pirates and, after 'hiding' the treasure, indicated with T, they must represent a 'mathematical map' to find it. Some examples (third and fourth year, primary school):

> The treasure is hidden in the square indicated with T. The landing point on the island is in 0. Represent the paths leading from 0 to T [18].

Some of the paths singled out by the pupils [19]:

- $0 + 11 \times 5 = 55$ [20]
- $0 + 10 \times 5 + 1 \times 5 = 55$
- $0 + 1 \times 5 + 10 \times 5 = 55$ [21]

> The treasure is hidden in the square indicated with A. The landing point on the island is in 99. Represent the paths leading from 99 to A.

- $99 - 11 \times 3$
- $99 - 10 \times 3 - 1 \times 3$
- $99 - 1 \times 3 - 10 \times 3$

Difficulty may be increased. For example, in order to reach the treasure in G starting from the landing point in 19, we can have:

- $19 – 1 \times 8 + 10 \times 6 = 71$
- $19 + 10 \times 6 - 1 \times 8 = 71$
- $19 + 9 \times 6 – 1 \times 2 = 71$

[17] *In order to clarify the instruction, we can recur to Brioshi, specifying that we must send him a path written in mathematical language.*

[18] *We can propose that the letter substitutes the 'number to be discovered'.*

[19] *Often, when counting how many times the rule is applied, pupils also insert the starting square. A collective test helps to solve the misunderstanding.*

[20] *Sometimes get so involved in the game that they propose very elaborate paths; this is a good occasion to reflect on the principle that more economic writings are to be preferred.*

[21] *The* ***didactical contract*** *can provide for a regular testing of the correctness of the expression:*

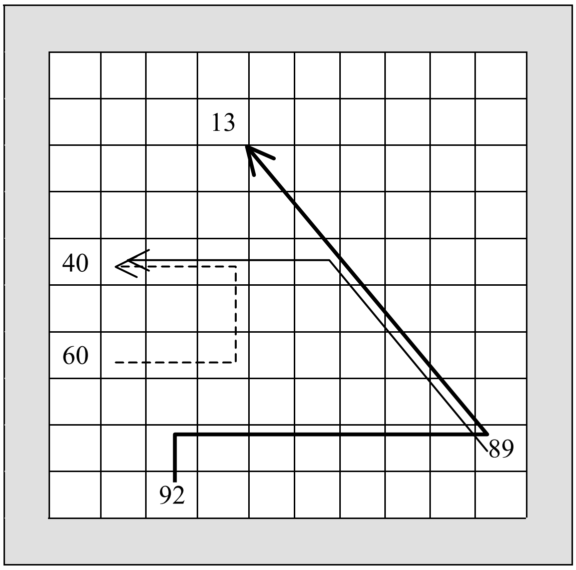
$0 + 1 \times 5 + 10 \times 5 = 55$
$0 + 5 + 50 = 55$
$55 = 55$.

We can underline the role of the ***equal sign*** *both as an* ***equivalence relation*** *and as an operator 'giving a* ***result****'. The equality as a relation of equivalence has a fundamental role in the building of algebraic thought.*

Suitable age related activities	*6*	*7*	***8***	***9***	***10***	***11***	*12*	*13*	*Comments*

B) This problem is the opposite of the preceding one.
This time the children identify themselves with the treasure hunters; they have to interpret the path written on a map and rewrite in mathematical language. The square with the first number of the expression is the ‘starting dock’.
For example: interpret the thick lined path and represent it.
A possible writing is the following:

$$92 - 10 \times 2 + 1 \times 7 - 11 \times 6$$ **[22]**



The class can test the correctness of the result later on:

$92 - 10 \times 2 + 1 \times 7 - 11 \times 6 =$ **[23]**
$= 92 - 20 + 7 - 66 =$
$= 13$ **[24, 25]**

[22] *Often pupils are puzzled by a writing, which does not end with an equal sign and with a result, outcome of calculations. It is a good occasion to underline the difference between represent and* ***solve****.*

[23] *It is better also to take advantage also of this situation to highlight the convention that the equal sign written at the end of the line is repeated at the beginning of the following line.*

[24] *The test can be done also in a way that privileges the point of view of the equivalence between two numbers, rather than the ‘classic’ one of the individuation of a result:*

92 - 10 × 2 + 1 × 7 - 11 × 6 = 13
92 - 20 + 7 – 66 = 13
13 = 13.

We can make the pupils reflect on the fact that the two expressions on the left and on the right of the equal sign represent different writings of the same number. *13 is the so called canonical form of the number, more synthetic, more ‘essential’, but more* ***opaque*** *as far as the meanings are concerned.*

[25] *With younger pupils calculations can be written down more slowly, highlighting the various steps; for example, in a third year (primary school, 8 year olds):*

92 - 10 × 2 + 1 × 7 - 11 × 6 =
= 92 - 20 + 7 - 66 =
= 72 + 7 - 66 =
= 79 - 66 =
= 13

We can also highlight the convenience of splitting up 66 in the second-last line and write:
= 79 - 60 - 6 =
= 19 - 6 =
= 13

Suitable age related activities	6	7	8	9	10	11	12	13	Comments

C) Activity with the aim of favouring the interpretation of the equal sign as an equivalence between writings, thus privileging the aspect of symmetrical property of the relation of equivalence (in fact it is a continuation of what was said in Comment 2 in this page).
Once again, children identify themselves with the treasure hunters. But this time they could not choose a single path, but they found two; pupils have the task of comparing them to understand if they both lead to the treasure [26].
For example the paths proposed for comparison are:

60 + 1 × 3 - 10 × 2 - 1 × 3 (dotted line) and 89 - 11 × 4 - 1 × 5 (thin line)

A group of pupils (fourth year, primary school, 9 year olds) elaborates this equivalence:

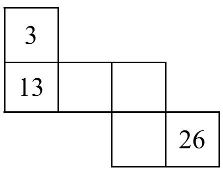
60 + 1 × 3 - 10 × 2 - 1 × 3 = 89 - 11 × 4 - 1 × 5
60 + 3 – 20 – 3 [27] = 89 – 44 - 5
63 – 20 – 3 = 45 - 5
43 – 3 = 40
40 = 40

Fourth phase

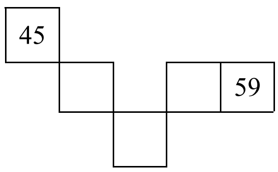
14. Activities with fragments of grid

We propose fragments of a grid [28]; pupils must complete them, writing numbers in the blank squares:

(a)



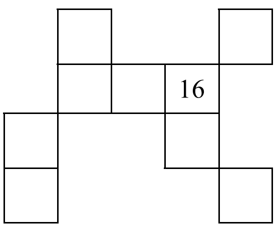
(b)



The pupils are invited to complete the squares describing aloud the regularities applied. For example:
- in (a) +1 is applied twice and then +10;
- in (b) +11 is applied twice and then -9.

Another exercise, slightly more difficult:

(c)



We ask the pupils to record the steps as they perform the calculations. For example, in (c):

16 – 1 → I write15	16 + 10 → I write 26
15 - 1 → I write 14	26 + 11 → I write 37
14 – 10 → I write 4	14 + 9 → I write 23
16 – 9 → I write 7	23 + 10 → I write 33

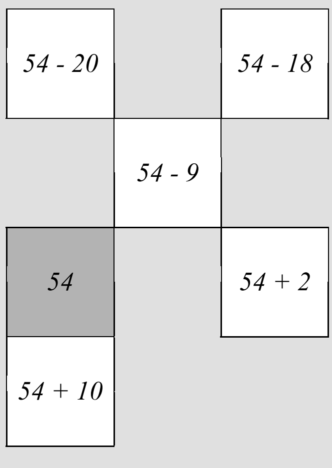
[26] *It is necessary to help the class understand that 'comparing' the two expressions means building with them and equivalence between writings. When calculations are finished, pupils might also find out that the equivalence does not exist and that the equal sign must be rewritten as '≠'.*

[27] *Cancellation might be applied here: in the second step (+ 3 and - 3) or even in the first one (+ 1 × 3) with (- 1 × 3), interpreting the sign as opposite operators.*

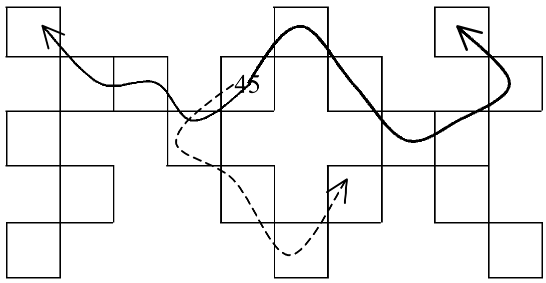
[28] *We suggest to hide the big grid so that pupils are forced to rebuild mentally its regularities. In case of difficulty, they can recur to the grid in the copybook or they can complete the grid as if it was inscribed in a rectangle (dotted in figure (a)).*

Expansion 3

It is advisable to accustom the pupils to represent the numbers of the fragment in function of a given number. This aspect has important repercussions in the algebraic field. In the example the reference is number 54:



Also in the following case, pupils must make the line of reasoning and the calculations explicit and then verify the result.



For example: start from 45 and

- follow an arrow and describe the path mathematically:

(a) thick arrow: 45 - 9 + 11 + 11 + 1 – 9 – 11 = 39
(b) dotted arrow: 45 + 9 + 11 + 11 – 9 = 67

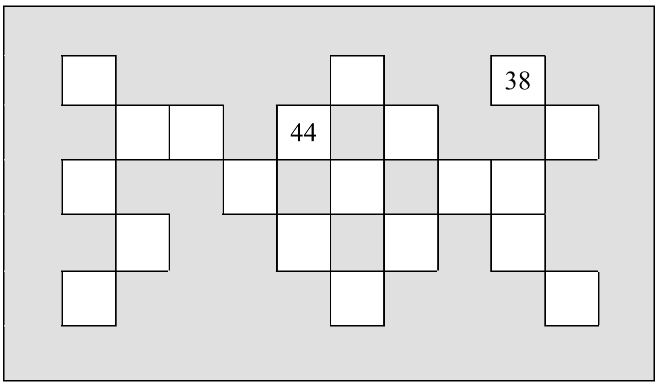
- propose some sequences and identify the paths.

For example: 45 + 9 – 11 – 1 – 11 = 31 [29] (thin arrow)

Fifth phase

15. The game of the Island

Rule: write the numbers in the blank boxes of the fragment, passing from one box to the other, but always staying on land (white boxes); if the rule is not respected, the player falls into the water (blue boxes and spaces).
The activity can be done collectively at the beginning, and every pupil makes the rule, which allows him to find the number to insert, explicit.[30]



In a second time, we make the pupils represent the paths and we activate comparison and discussion.
For example, the path from 45 to 39 can be described as follows:

45 - 9 + 11 +11+ 1 - 9 - 11 = 39

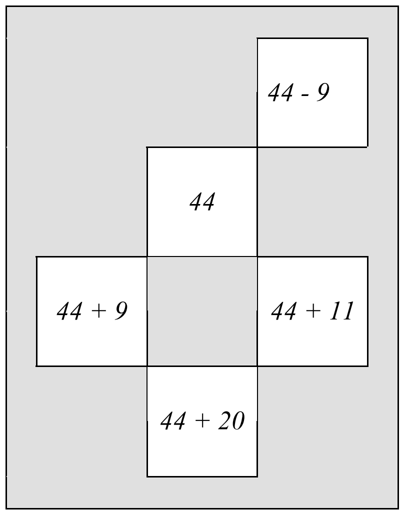
which is equivalent to:

45 + 11 × 2 - 9 + 1 - 9 + 1 -11 = 39

[29] This is an occasion to reflect on equivalent writings more convenient than the ones proposed, for example:
(a') 45 – 9 + 11 × 2 + 1 – 9 – 11
or on cancellations, for example:
(a) 45 - 9 + 11 + ~~11~~ + 1 – 9 – ~~11~~ = 39
(b) 45 + ~~9~~ + 11 + 11 – ~~9~~ = 67

*[30] When completing the fragment, pupils may be asked to represent the numbers in a non-canonical form in function of that of the starting island (see **Expansion** - preceding page).*

In this case, for example:



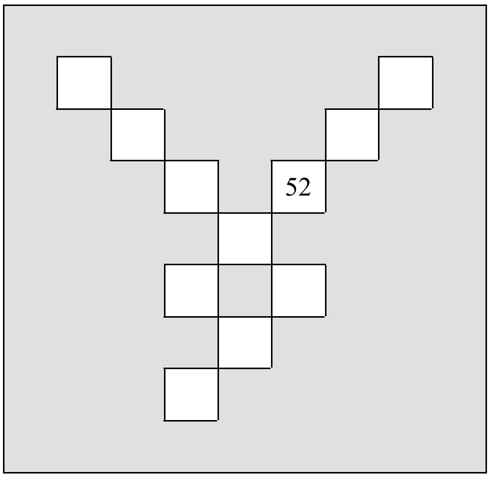
In a third year (primary school) a little girl proposed that the number might be the 'name of the island'. From this idea, games like the following originated: «Who can put the island in its right place, called '25 minus 9 minus 11'?». As usual, the collective comparison of the strategies adopted by the pupils to identify the corresponding island is very important.

ArAl Project	U2. Numbers grids

Suitable age related activities	6	7	8	9	10	11	12	13	Comments

16. The Never Never Island

There can also be some *fantasy* islands, like the one reproduced in the following drawing. The comparison with the starting grid clearly highlights the reason why a name coming from Peter Pan was chosen: in fact, we discover that the fragment cannot be a part of an original grid 0 - 99, in which number 52 is in the third column. [31]



[31] Before realising the situation, pupils often elaborate some 'mechanical' strategies to complete the 'impossible' squares of the Never Never Island.

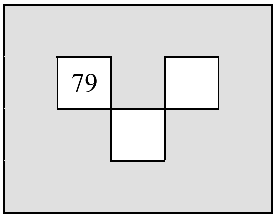
For example, in the situation we are examining, the two squares on top left, which in fact cannot belong to the grid*, are often filled in with the numbers 28 and 39. In this case, if on the one hand it is not clear to the pupils that the playground is the grid 0-99, it is also true that, coherently to the rules, pupils put number 50 on the left of number 52. Moreover, they continue to subtract 11, thus obtaining numbers 39 and 28.*
In other solutions, pupils alter the playground by eliminating or adding lines or columns to fit it to a given problem.

The discussion helps to overcome the obstacle.

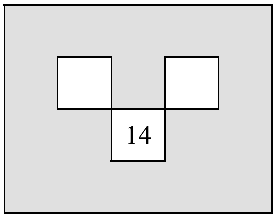
Suitable age related activities	6	7	**8**	**9**	**10**	11	12	13	Comments

17. To reinforce comprehension we propose more example with islands; pupils have to understand whether these are ‘existing’ islands or not. Here are some examples:

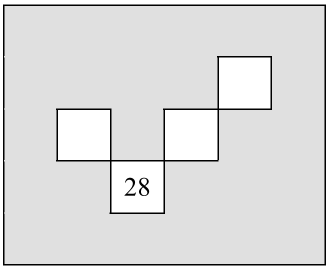
(a)



(b)



(c)



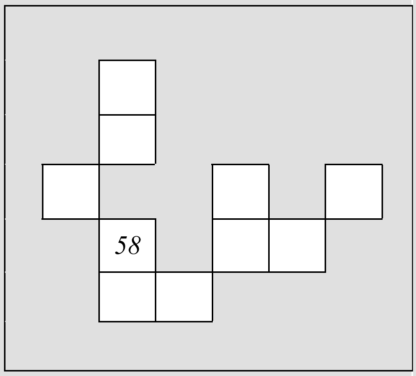
(a): by applying the rule ‘+ 11’ number 90 should be inserted and, by applying the rule ‘- 9’, we should insert 81 in the third square. But this is not compatible with our grid, hence (a) is a ’’Never Never Island’.
With similar arguments we find out that (b) is a ‘Never Never Island’, but the island (c) is an ‘existing island’.

Note 1

Often in situations of exploration, discoveries and intuitions might come about, which were not predicted when preparing the activity.
In a third year (8 year olds), a little girl realised that calculations were not necessary to understand whether the island existed or not. She explained that she had observed that the numbers in the last right column of the grid end with number 9. Consequently, as in (a) after number 79 there were other columns, without any calculation, she concluded that it was impossible for the island to be compatible with the grid.
This remark led to the formulation of some regularities regarding perimetrical lines and columns of the grid; the following definitions were the final result of a patient work of collective building of knowledge:

- *Numbers on the right border have 9 as a unit.*
- *Numbers on the left border have 0 as a unit.*
- *Numbers on the bottom border have 9 as a decade.*
- *Numbers on the top border go from 0 to 9.*

These conclusions can be used to set up test problems. Pupils must justify compatibility of a fragment with the grid, arguing on the basis of such conclusions.
In the example, as the last right column of the original grid contain numbers, which have 9 as a digit for units, it is not possible to have four squares on the right of number 58.

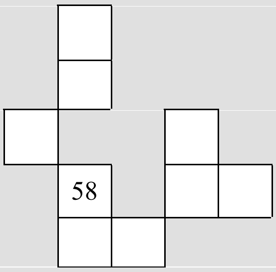


ArAl Project	U2. Numbers grids

Suitable age related activities	6	7	8	9	**10**	11	12	13	Comments

Supplementary activity 1

We start off proposing an impossible fragment; pupils explore and argue its incompatibility with the grid (in the following example, as the grid ends with numbers having 9 as a digit of the units, it is impossible to have three squares on the right of number 58).



Later on we pass on to search under what conditions the examined fragment can be compatible with the grid 0-99.
The problem can be formulated as follows:

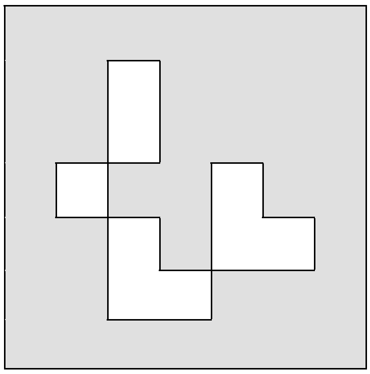
> What numbers of the grid 0 – 99 can be put instead of number 58 for the problem to have a solution?

The exploration can be conducted in an experimental way, with different attempts [32]. For example, we discover that number 14 or number 92 don't fit in, whereas with numbers like 51 or 76 the problem can be solved, and so on. As explorations go on, pupils tick the squares not allowing a solution.
At the end we find out that:
- We can put numbers not ending with 7 - 8 - 9 – 0;
- We cannot put numbers bigger than 86;
- We cannot put numbers smaller than 31;
- To conclude: the digit for units must be included between 1 and 6, the digit for decades between 3 and 8.

0	1	2	3	4	5	6	7	8	9
10	11	12	13	14	15	16	17	18	19
20	21	22	23	24	25	26	27	28	29
30	31	32	33	34	35	36	37	38	39
40	41	42	43	44	45	46	47	48	49
50	51	52	53	54	55	56	57	58	59
60	61	62	63	64	65	66	67	68	69
70	71	72	73	74	75	76	77	78	79
80	81	82	83	84	85	86	87	88	89
90	91	92	93	94	95	96	97	98	99

To conclude, the number, which can substitute 58 is one included in the square having as vertex numbers 31, 81, 86, 36.

[32] *It can be useful to apply some devices to help pupils in the search. We suggest two:*
- *report on a paper for overhead projector (or on a transparent paper, anyway) the fragment, then place it over the grid 0 – 99 and move within the grid, in order to find the part of the grid compatible with the grid.*
- *prearrange a paper with the shape of the fragment cut out and with a highlight on the square containing number 58 (in the jargon of paper modelling this instrument is called 'mask'). By placing the mask over the grid, it is possible to identify easily the numbers that solve the problem*



Suitable age related activities	*6*	*7*	*8*	***9***	***10***	*11*	*12*	*13*	*Comments*

Expansion 4

The observation of the grid can lead to interesting discoveries.

	I	II	III	IV	V	VI	VII	VIII	IX	X
I	0	1	2	3	4	5	6	7	8	9
II	10	11	12	13	14	15	16	17	18	19
III	20	21	22	23	24	25	26	27	28	29
IV	30	31	32	33	34	35	36	37	38	39
V	40	41	42	43	44	45	46	**47**	48	49
VI	50	51	52	53	54	55	56	57	58	59
VII	60	61	62	63	64	65	66	67	68	69
VIII	70	71	72	73	74	75	76	77	78	79
IX	80	81	82	83	84	85	86	87	88	89
X	90	91	92	93	94	95	96	97	98	99

We start by numbering the lines and columns of the grid.
Then we invite the class to analyse a number contained in the grid and to identify the relation between its digits and the numbers and of the line and of the column, on whose crossing the number is placed.
For example, for number 47 pupils make their analysis explicit in this way:

- the digit of *tenths* is represented by *the number of the line minus 1*;
- the digit of *units* is represented by *the number of the column minus 1* [33].

Similarly, for other numbers we find out that:

- 76 is placed at the crossing between line VIII and column VII;
- 25 is placed at the crossing between line III and column VI;
- 36 is placed at the crossing between line IV and column VII;

and so on.

We can then proceed with the opposite exercise: find the number at the crossing between line VIII and column IV (number 73).

Similarly:

- the number at the crossing between line V and column IX is 48;
- the number at the crossing between line II and column X is 19;
- the number at the crossing between line VI and column III is 52;

and so on.

This **Expansion** can lead to a deep study on subjects like:
(a) the meaning of quotient and remainder of a division, as numbers on the same line divided by 10 give the same result and numbers on the same column divided by 10 give the same remainder;
(b) the polynomial form of the number;
(c) the numeration systems on basis different from10, modifying the dimensions of the grid.

[33] *The reason of this feature is that the first number of the first line is 0 and under that we find 10, 20, 30 e 40 (in the fifth line). The same thing is valid for the digit representing units: 0 tenths correspond to the first line, 0 units correspond to the first column.*

Expansion 5

For intermediate school: the number at the crossing between line x and column y has x - 1 as a digit for tenths and y - 1 as a digit for units. Thus, we can represent as

$$(x - 1) \times 10 + y - 1.$$

The situation gives a hint for reflection on the number and its polynomial form.

Sixth phase

18. Towards a generalisation

We analyse what happens when the dimensions of the grid are modified; for example a 0 – 63 grid formed by 8 lines and 8 columns.

0	1	2	3	4	5	6	7
8	9	10	11	12	13	14	15
16	17	18	19	20	21	22	23
24	25	26	27	28	29	30	31
32	33	34	35	36	37	38	39
40	41	42	43	44	45	46	47
48	49	50	51	52	53	54	55
56	57	58	59	60	61	62	63

In this new situation we examine again the rules we studied in the 0 – 99 grid (10 lines and 10 columns); we find that:

- along a line: +1 and –1;
- along a column: +8 e –8;
- along an oblique NE-SW: +7 e –7;
- along an oblique NW-SE: +9 e –9.

The rule has remained unchanged along the line; it is different in the other three cases.

We propose now a 7 × 7 grid…

0	1	2	3	4	5	6
7	8	9	10	11	12	13
14	15	16	17	18	19	20
21	22	23	24	25	26	27
28	29	30	31	32	33	34
35	36	37	38	39	40	41
42	43	44	45	46	47	48

… and a 5 × 5 grid:

0	1	2	3	4
5	6	7	8	9
10	11	12	13	14
15	16	17	18	19
20	21	22	23	24

ArAl Project	U2. Numbers grids

Suitable age related activities	*6*	*7*	*8*	***9***	***10***	***11***	***12***	***13***	*Comments*

In this case rules are:
- along a line: +1 e –1;
- along a column: +5 e –5;
- along an oblique NE-SW: +4 e –4;
- along an oblique NW-SE: +5 e –5.

The rule remained the same only along the line; it is different in the other three cases.
We explore other cases and we sum up the results of the enquiry in a board; by studying it we find the **general case**. For example:

shift	10 × 10	8 × 8	7 × 7	5 × 5	4 × 4	**n × n**
→	+ 1	+ 1	+ 1	+ 1	+ 1	**+ 1**
←	- 1	- 1	- 1	- 1	- 1	**- 1**
↓	+ 10	+ 8	+ 7	+ 5	+ 4	**+ n**
↑	- 10	- 8	- 7	- 5	- 4	**- n**
↙	+ 9	+ 7	+ 6	+ 4	+ 3	**+ n - 1**
↗	- 9	- 7	- 6	- 4	- 3	**- n + 1** [34]
↘	+ 11	+ 9	+ 8	+ 6	+ 5	**+ n + 1**
↖	- 11	- 9	- 8	- 6	- 5	**- n - 1**

[34] *The* naive *discovery of the operator '- n + 1' had a story, in a class of fourth year (9 year olds), which deserves to be told.*
The compilation of the grid had been faced individually at first and then the various solutions had been compared and commented – as always - collectively.
The discussion had outlined two different strategies in identifying the operator corresponding to the arrow ↗ in the column of the ***n*** × ***n*** *grid:*
(a) some pupils had realised that it was sufficient to invert the operator +n-1, corresponding to ↙ (identified beforehand), thus obtaining the operator - (+ n - 1);
(b) on the other hand some other pupils had solved the problem by moving in a non-diagonal direction: first along the direction ↑ (operator – n) and then along the direction → (operator +1), thus obtaining the operator -n+1.
The comparison between the strategies and the observation that both were correct allowed to conclude that they were equivalent and to represent the equivalence for Brioshi:

- (+ n - 1) = - n + 1.

Once again, the activity highlights how stimulating situations favour the development of ***algebraic babbling****.(even in unpredictable ways, not considered when planning the activities)*

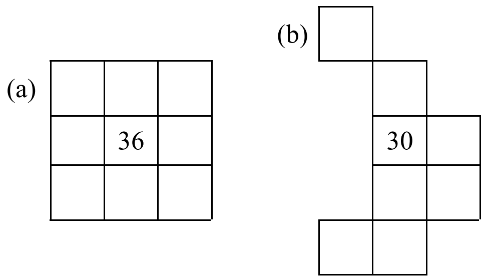
With older pupils we can highlight the equivalence of the writings:

$$-n + 1 = -(n - 1)$$
$$-n - 1 = -(n + 1)$$

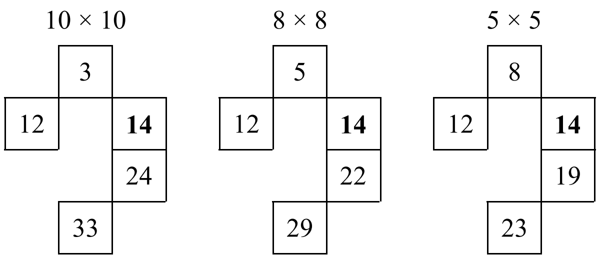
They can draw this conclusion not through physical movement on the grid, like younger pupils did, but comparing the values on the grid line by line and identifying the link between the dimensions of the grid and the operator corresponding to the arrow. This strategy is more complex because it is developed in a more abstract environment.

Suitable age related activities	6	7	8	9	10	11	12	13	Comments

19. Some exercises are done in order to reinforce the achievements of situation **18**.
Two examples referring to a fragment of a grid 8 × 8. Obviously the pupils can help themselves referring to the preceding table.



20. We can complete the same fragment supposing it belongs to grids of different dimensions. Some examples (the only number initially present is 14 **35**):



35 *It is necessary to pay attention to the number initially inserted in the fragment, because it's easy to meet a 'Never Never Island'.*

36 *Because along the oblique lines we can find the regularity +6. By adding twice number 3, we add 6, that is +3+3 = +3·2= +6.*

37 ***No**. 28 would be coloured: we can try different ways or build a numerical succession (... 16, 19, 22, 25, 28).*

38 *No. We might reason in different ways, For example: (i) by applying the rule + 3, the sequence would be 25, 28, 31, 34, 37, 40, 43 and number 41 does not appear; or (ii): we notice that, in the last column, the step from 4 to 19 is 15. Hence, we can suppose this succession: 4, 19, 34, 49. This means the box number 46 is coloured because 46 = 49 – 3.*

39 *Number 46 is coloured because 46 = 43 + 3.*

Supplementary activity 2

We begin by assigning this task: in a particular grid - with a fixed width and infinite length- we colour the boxes containing the number which can be obtained starting by 1 and adding the same number every time.
In the example proposed the grid is 5 of width and the added number is 3.

0	1	2	3	4
5	6	7	8	9
10	11	12	13	14
15	16	17	18	19
20	21	22	23	24

The exploration can now move in different directions:

- Why is the colour distributed in this way in the boxes? **36**
- Would number 29 be in a coloured box? **37**
- And number 41? **38**
- And number 46? **39**

Expansion 6

The highlighted numbers in the grid with width 5 are those which, divided by 3, give 1 as a remainder. For more advanced classes they can be expressed as 3n + 1, with n natural number, and this might give the chance to face the study of congruences and the classes of remainders. From this point of view, we can state whether a box is coloured or not. In our example:

29 : 3 = 9 with remainder 2
41 : 3 = 13 with remainder 2
46 : 3 = 15 with remainder 1.

46 is the only 'coloured number' because it is the only one that gives 1 as a remainder, when divided by 3.
Another generalisation can be drawn starting from any a*, and adding any* b

ArAl Project	U2. Numbers grids

Suitable age related activities	*6*	*7*	*8*	*9*	***10***	***11***	***12***	***13***	*Comments*

Seventh phase

21. We resume the activities already carried out; pupils represent the number of a grid *in function of a given number and of the dimensions of the grid*. For example: write the missing number in a 3×3 grid in function of the number in the central box.

0	1	2
3	4	5
6	7	8

	4	

Pupils can repeat the same situation in a bigger grid; for example a 6 × 6 grid.

22. The following step is that of making the pupils write some numbers in function of any number *a* of a grid 10 × 10.

	a	

Then we can test how, by substituting *a* to a value, we can obtain any part of the original grid (obviously *a* cannot belong to the 'frame' of the grid; in this case the fragment would be a 'Never Never Island').
The conclusion is that, within the same grid, the relationships among the numbers remain unchanged, whatever part of the grid we consider.

Solutions:

21.

4 - 4	*4 - 3*	*4 - 2*
4 - 1	*4*	*4 + 1*
4 + 2	*4 + 3*	*4 + 4*

26 - 7	*26 - 6*	*26 - 5*
26 - 1	*26*	*26 + 1*
26 + 5	*26 + 6*	*26 + 7*

22.

a - 11	*a - 10*	*a - 9*
a - 1	*a*	*a + 1*
a + 9	*a + 10*	*a + 11*

Suitable age related activities	*6*	*7*	*8*	*9*	*10*	***11***	***12***	***13***	*Comments*

23. Activity with an irregular fragment of the 0-99 grid.
In a box we put a number *a*; pupils must complete the blank boxes in function of *a*. An example:

	a	

23. *Solution*

a - 11		*a - 9*
	a	*a + 1*

24. We propose irregular equivalent fragments of the 0-99 grid, in which *a* is placed in different boxes. Pupils must complete the fragments.

(1)

	a	

(2)

		a

Once the problem has been solved, we can look for the values that would appear in the blank boxes among those of the fragment (we might say that pupils must recover the ‘internal consistency’ of the grid). In (1) we might verify that, for example, among the boxes containing numbers ‘a’ and ‘a + 20’ a box containing ‘a + 10’ is hidden and that, below the box containing ‘a + 11’, and on the right of ‘a + 20’, the number ‘a + 21’ is hidden

24. *Solution*

a - 1	*a*	*(1)*
		a + 11
	a + 20	

a - 12	*a - 11*	*(2)*
		a
	a + 9	

25. In open order some boxes of the 0-99 grid are assigned, which contain some numbers expressed in function of a; pupils must compose them in a coherent fragment using the rules of the grid.

a	a + 10	a - 20	a + 2	a - 18	a - 9

25. *Solution*

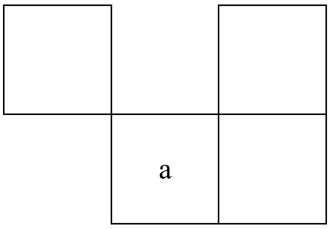
a - 20		*a - 18*
	a - 9	
a		*a + 2*
a + 10		

Suitable age related activities	*6*	*7*	*8*	*9*	*10*	***11***	***12***	***13***	*Comments*

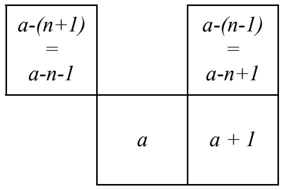
Eighth phase

Next activities, analogous to those of the **situations 23-25**, apply to the **n** × **n** grid (we assume that the problem is solvable, which means that *a* has been chosen so that the fragment turns out to be within the **n** × **n** grid).

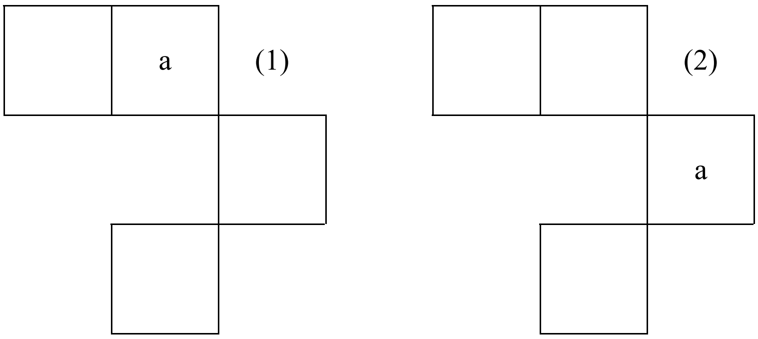
26. (see Situation **23**)



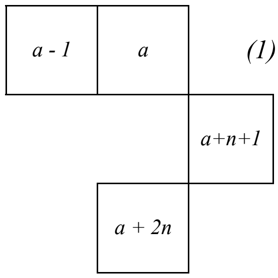
26. *Solution*

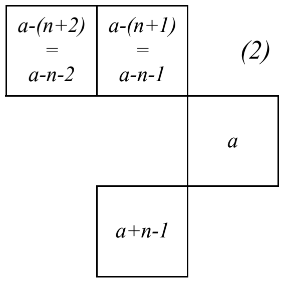


27. (see Situation **24**)



27. *Solution*

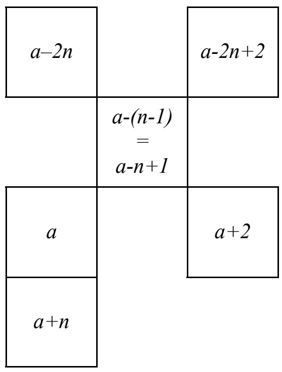




28. (see situation **25**)

a	a+n	a–2n	a+2	a-2n+2	a–n+1

28. *Solution*



It is better to reflect on these equivalences:

- a – n + 1 = a – (n – 1)
- a – 2n + 2 = a – (n – 1) × 2 = a – (2n – 2)

This activity favours also awareness in the interpretation of writings, which differ from the **syntactic** point of view, but are equivalent from the **semantic** one.

29. We start again from situation 20. We complete a fragment, in which a number has been written and put into grids of different dimensions.
For example:

(a) 10 × 10
(b) 8 × 8
(c) 5 × 5

	13	

***29**. Solutions:*

(a) 10 × 10

2	3	4
12	13	14
22	23	24

(b) 8 × 8

4	*5*	*6*
12	*13*	*14*
20	*21*	*22*

(c) 5 × 5

7	*8*	*9*
12	*13*	*14*
17	*18*	*19*

Expansion 7

If we propose a 7 × 7 grid, the problem becomes impossible (pupils might recognise a 'Never Never Island').
Therefore, this problem has a solution only if we consider grids of particular dimensions.
Let's take the generic n × n grid into consideration.
It is interesting to study, for what value of n the problem has a solution.
The box containing number 13 is the central one: It is necessary to make it figure on the frame of the n × n grid.
$\boxed{13}$ cannot stay on the first line of the grid. Such line contains the numbers from *0* to *n - 1*, hence it must be: $13 > n - 1$
$\boxed{13}$ cannot stay on the last line of the grid. Such line contains the numbers from $n^2 - n$ to $n^2 - 1$, hence it must be: $13 < n^2 - n$
$\boxed{13}$ cannot stay on the first column of the grid. Such column contains the multiples of n, and 13 cannot be a multiple of n: the remainder of the division of 13 by n cannot be 0.
$\boxed{13}$ cannot stay on the last column of the grid. Such column contains the numbers, which divided by n give *n-1* as a remainder: the remainder of the division of 13 by n cannot be *n - 1*.
To sum up, the conditions which make it possible to solve the problem in n × n are the following:
a) $n - 1 < 13 < n^2 - n$
besides, being defined as R the remainder of the division of 13 by n, it must be:
b) $0 < R < n - 1$
Under such conditions, the problem can be solved.

Expansion 7

Solution of the problem:

n × n

11-n-1	*13-n*	*13-n+1*
13-1	*13*	*13+1*
13+n-1	*13+n*	*13+n+1*