INFLUENCES OF A PROCEDURAL VISION OF ARITHMETICS IN ALGEBRA LEARNING

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We trace the outline of the ArAl Project, which promotes didactic innovation in primary schools, aimed at an early approach to algebra. The complex structure of this project entails the involvement of teachers in the discussion and reflection on the results of the experiment, making it at the same time a teacher-training project. Here we focus our attention on the influences of a procedural vision of arithmetics in pupils, which conditions their behaviour when confronted with algebraic tasks. We reflect on the teachers' role in this context, and highlight the need for change in the teacher culture.

Theoretical Framework

Over the past twenty years, research has focussed on a large number of possible approaches that increase the *meaning* of algebraic processes and objects (Arcavi, 1995). Some of the principal forms are: *problem solving* (where emphasis is given to the analysis of problems and equations); the *functional* approach (the use of letters to indicate measurement and the formal coding of relations among measurements); the *generalization* approach (the use of expressions to represent geometric patterns, numerical sequences, "rules"). A determining role is attributed to the *linguistic* approach and to research that faces the didactical developments starting from the *concept of algebra as a language* (Arzarello et al., 1993; Malara and Iaderosa, 1999; Radford, 2000; Menzel, 2001). This role becomes even more significant if it is associated with the hypothesis of an early approach to algebraic education beginning from the didactical revision of the relations between arithmetics and algebra. Many researchers indeed have underlined that the difficulties in the approach to algebra are rooted in the scarce attention paid to the relational or structural aspects of arithmetics, which constitute the basis of elementary algebra (see the Kieran surveys, 1990, 1992; Linchevski 1995). In recent years, many studies have been developed, operating from various stances and with different approaches to the matter (Ainley 1999; Brawn and Coles, 2000; Carpenter and Franke, 2001; Carraher et al., 2000, 2001; Da Rocha Falcão, 1997, 2000; Kaput and Blanton, 2001, 2002; Savadosky, 1999); within this framework and on the basis of our own previous studies, we have developed the ArAl Project, which was conceived for the renewal of the teaching of arithmetics in the sense of early algebra (Malara and Navarra, 2001, 2002; Malara et al., 2002).

Our Hypotheses

The ArAl project is based on the hypothesis that the main cognitive obstacles in learning algebra are to be found in the pre-algebraic field, and that many of these spring up from unsuspected arithmetical contexts and they then become conceptual obstacles to the development of algebraic thinking, because of the weak conceptual control which many students have over the *meanings* of algebraic objects and processes. In the perspective of a conception of algebra as a language, the hypothesis of the ArAl Project is that there exists a strong analogy between the methods for learning a spoken language and the algebraic language.

In order to explain this point of view, we shall employ the "*babbling*" metaphor. When a child learns a language, he or she masters the meanings of words and their supporting rules little by little, developing gradually by imitation and self-correction, right up to the study of the language at school age, when the child begins to learn to read and reflect on the grammatical and syntactic aspects of the language. Traditionally, in the teaching of the algebraic language, one starts with the study of the rules, as if the formal manipulation should have precedence over the understanding of meanings.
The syntax of algebra therefore tends to be taught while overlooking its semantics. The mental models for algebraic thinking should be built rather more in an arithmetical framework - beginning from the very first year of primary school - through initial forms of *algebraic babbling*, teaching the pupil *to think of arithmetic in algebraic terms*, in other words, building her/his algebraic thinking *progressively into close interlacement with arithmetic*.
To this end, it is necessary to construct an environment which stimulates even informally the autonomous elaboration of the algebraic babbling and therefore the playful, experimental and continually redefined acquisition of a new language in which the *rules* may find their place just as gradually within a didactical context which is tolerant of initial syntactically "shaky" moments.
The teacher's role in the management of such an environment is made more difficult by the fact that almost all compulsory school teachers do not have a mathematical university background, as the majority come from areas of humanistic and pedagogical education (primary teachers) and scientific education (junior secondary teachers).
Their cultural background and beliefs naturally influence their choice of ways through which they impart pupils basic knowledge. The Project activities and the analysis of what happens in class during their implementation highlight these influences, and allow us to formulate hypotheses for their (not easy) possible overcoming.
The situations laid down in each Unit take place in stimulating didactical environments – which can often be difficult to handle – and require of the teacher various competences and subtle abilities. In other words, teachers who intend to embrace innovative teaching approaches must be prepared to combine, as previously mentioned, their existing knowledge, competences and beliefs with a mix of far from marginal methodological and organizational aspects (to stimulate activities with a high metacognitive content, to favour the reflection on language, to promote verbalization and argumentation, to reach a fine analysis of the protocols). All of these aspects operatively support an actual *culture of change*.

Methodology

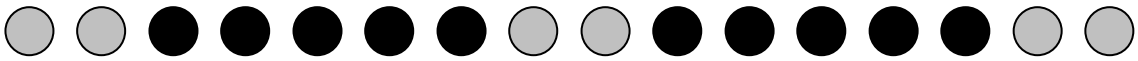
Since 1998, the ArAl Project, with its 63 teachers e more than 2000 pupils involved, is put forward as an integrated training system. Through seminars and investigation meetings, teachers contribute to a first drafting of the materials to be tested in the classroom, to the recording of experiments (even with the use of the recorder) and the periodical assessment of activities.

At the beginning of each school year, the Project's leaders propose some themes, around which the experimentation is organized. In 10 or 12 of the almost 70 classes involved in the Project, the classroom activity is led by "tandem partners", i.e. a researcher from the team and the class teacher. Each year some 140 hours of this tandem work are completed. Minutes are kept for each of these joint classes and the 140 minutes - duly analysed and integrated with each other - represent the control basis for the existing teaching Units, and for the writing of new ones.
During these joint classes, metacognitive and metalinguistic activities are promoted: the verbalization of pupils, argumentation, the collective comparison of protocols (for more details on this point see Malara & Navarra 2002) .
We shall now take a look at two examples - the first deals with the search for regularities, the second with the distributive property - and examine how the teachers' cultural background (their knowledge of mathematics is generally poor) can inhibit some excellent opportunities for algebraic babbling.

Examples

1) Unit: ***Regularity: Frames and Necklaces***

The aim of this Unit is to explain how the study of *regularity* within a sequence can lead to the discovery of a *key for the algebraic reading* of its structure, and thus to its r*epresentation* in a formalised language. Algebra thus not only becomes a language used to describe reality, but it also improves the *comprehension* of such reality. But the origin of this pathway is based on the initial relationship, which is established between the observer and the pattern in question, and it is on this particular aspect that we wish to dwell.
Within this Unit, pupils must analyse a series of frames and necklaces, which contain various alternating sequences of grey and black pearls. Example:



The task requires the pupils to identify the colour of the pearl found in a given position, with an ever-increasing ordinal number referred to the pearl's position. Individual pupils propose their own hypotheses, which are then discussed collectively with the rest of the class. In the example, the questions asked are:

What is the colour of the 12^{th} pearl? And the 35^{th}? And the 123^{rd}?

There is obviously a huge difference between those positions where the pearls *can actually be seen*, and having to *imagine* pearls within an infinite necklace. Indeed, the increasing number makes it necessary to adopt a strategy, because starting already from the 35^{th}, the pearls are then no longer visible. The initial phase is often difficult for both pupils and teacher, usually not accustomed to working with regularity searches (a synthesis of classroom discussions is provided in Unit 5 of the ArAl Project). Our analysis of the attitude of both pupils and teachers has brought the following considerations.
The basic obstacle is to be found in the *initial perception* of the necklace: the spontaneous perception of the majority of interviewees usually catches the *alternating pat-*

tern of groups of pearls of varying number and colour, and the search for a hidden pearl is lost in this infinite repetition. *Many* regularities are indeed discovered, but they become - so to speak - unproductively "fragmented" between black and grey pearls, and between the number of the former and latter. This fragmentation stops the development of exploration and inhibits the identification of the *structure* of the necklace, i.e. of the network of *relationships* existing between the various pearl components.

Regularity only begins to be identified once there is a *breakaway from this perception*, as a result of a *metacognitive* type of process: that is to say, the observer must process the information given by such a perception, *changing completely his own point of view*. In terms of Gestalt psychology, the idea would be expressed in this way: it is necessary to *restructure the field*.

This restructuring is far from being a simple operation. The identification of regularity appears to take place in two distinct ways: either it is instinctive, or the result of a cultural acquisition, that is to say, the operator has to *learn what he must search for*. In the first case, the observer spontaneously excludes all *distractions* in selecting the module that is *appropriate for the particular objective of this activity*. In the second case, the observer is attracted by a flood of perceptions, which derive from a continual spontaneous restructuring of the field, which leads to the identification of variously aggregated repetitions, symmetries and groupings, which nevertheless only constitute an indiscriminate and apparently infinite collection of possibilities of interpretation, without containing a single interpretation which proves to be *dominant* above all others. We could say that the *aesthetic* aspect dominates over the *logic* aspect.

These are, nonetheless, the most stimulating and productive situations, since they lead to a search for strategies that can help the observer to identify a *filter* for his or her perceptions. One strategy which is proving very useful in the search for regularity in a model such as the necklace is the use of a unidirectional *frame* and related *module* (normally referred to as "template" in the Italian primary school environment). The exploration of the relation between frame and module enables the capture of the *reading key* for the succession, followed - with varying characteristics and difficulties - by the identification of the *structure* and its *representation* in a formal language.

As a partial conclusion, we could say that - in this particular perspective - the difficulty for the teacher is not originally of a mathematical nature, but rather due to a lack of familiarity with approaches that he or she may have seldom had the opportunity of exploring during their professional career. The correct reading key for the *perception* of regularity makes it possible to *tackle the problem* in a productive way. Difficulties of a mathematical kind naturally also appear, presenting just as many challenges, which can only be understood as such if the problem situation has become significant for the observer - i.e. *transparent* - also thanks to the *mediator*, which has made it possible to ferry the thought across to the *mathematical shore* of the problem. It seems obvious that the teacher who has not acquired a productive mental frame for these particular situations will tend to avoid them, out of insecurity, actually stopping when confronted with difficulties which precede the mathematical

ones, thus missing precious opportunities for creating - even in very young pupils (7-8 year-olds) - stimulating embryonic forms of pre-algebraic thinking.
Once they have acquired the correct reading key for the pattern, observers penetrate its structure and are gradually lead towards its representation in mathematical language. An analysis of this new teaching scenario can help us highlight another point where the teacher's lack of familiarity with relational thinking inhibits the creation of algebraic babbling, and thus the development of thought towards generalisation.
A class (4th year primary school, 9-year-old pupils) is able to analyse the following frame, whose module consists of 7 elements: 4 stars and 3 half-moons.

…

The first questions asked are: which is the 7th symbol and which is the 11th?
It is easy to see that the 7th symbol is a moon and the 11th a star. Finding the 46th symbol, however, is much more complex. One has to understand the necessity of dividing 46 by the number of elements in the module, i.e. 7. Performing the division, we get a quotient 6 and a remainder 4, where 6 represents the number of modules and 4 the position of our element within the module. Thus, the element in question is in fourth position within the seventh module, therefore it is a moon. In the same way, we discover which element is in 1999th position: dividing 1999 by 7 we get a quotient 285 and a remainder 4, thus our element is in 4th position of the 286th module: i.e. a star.
The significant number, therefore, *is not the quotient but the remainder*, and this is not a trivial discovery for the observers, both teachers and pupils. Pupils would certainly be facilitated in their search, if they were accustomed to seeing division as a binary operation, i.e. an operation on *a pair* of numbers (the dividend and the divisor), which results in *another pair* of numbers - quotient and remainder - and if they were able to express in several ways the relationship between dividend, divisor, quotient and remainder. Example:

$46 = 7 \times 6 + 4$ $\qquad$ $46 - 7 \times 6 = 4$ $\qquad$ $(46 - 4) : 6 = 7$

With older pupils, it is possible to aim at generalisation: given two natural numbers a and b, where $b > 0$, and having defined q and r respectively as the quotient and remainder of the division a : b, we can express the relationships between them in several ways: $a = q \times b + r$ $\qquad$ $a - q \times b = r$ $\qquad$ $(a - r) : b = q$.
These transcription exercises, if carried out habitually from the very first approach to division, could help pupils avoid rigidity in the formal coding of the relationship between the various division elements, encouraging later flexibility in performing algebraic translations. Instead, teachers generally privilege the representation in columns, linked to the processing of the algorithm, which focuses the attention on the quotient and neglects the remainder. Of course, writings such as "46 : 7 = 6 with remainder 4" - which put on the same level quotient and remainder - can be considered an interesting pupil attainment (the importance of representing in various ways the relationships between dividend, divisor, quotient and remainder has now been stressed for some time, as documented in Doblaev, 1969, and recommended even recently by Savadosky, 1999).

2) Unit: *The Distributive Property*

During the preparation of material relating to the theme of *Distributive Property*, teachers (participating in the Aral Project) and 5th year primary and 1st junior high pupils (not participating) are presented with some problems with the aim of producing some forecasts of the results of introducing the materials into the classroom.

One of the proposed problems is the following:

Leo's shopping

Leo has bought himself for school 7 notebooks and 3 pens. Both notebooks and pens cost ¤ 1.5 each. Represent how much money Leo has spent.

These are some types of expected correct solutions: a) 1,5 x 7 = 10, 5 ; 1,5 x 3 = 4,5 10,5 + 4,5. b) 1,5 x 7 + 1,5 x 3; c) 1,5 x (7 + 3). d) (7 + 3) x 1,5.

It must be pointed out that it is not necessary to find any results.

Three typical solutions emerge:

1) Age: 11 years

The pupil immediately says that he is used to finding partial results, and that he becomes otherwise confused. He makes all his calculations in columns:

1,5 x	1,5 x	10,5 +
. 7	. 3	. 4,5
10,5	4,5	15,0

2) Age: 12 years

This female pupil too makes the partial calculations, but she uses a horizontal representation; the last expression lacks an equals symbol and a result: 1,5 x 7 = 10, 5 ; 1,5 x 3 = 4,5 ; 1,5 x 7 + 3 . The third writing is corrected, with some input from the teacher, into the following one: 1,5 x (7 + 3).

During the discussion, though with some hesitation, this pupil recognises the possibility of applying the distributive property.

3).Teacher: (7 + 3) x 1,5

An analysis of these protocols leads to some observations.

The 1) protocol shows a decidedly *procedural* approach: there is not only a need for the certainty given by the result, but also the chosen *representation* reveals the pupil's (perhaps the teacher's too?) anxiousness not to lose control of the operation's correctness. The *separate* operations and the lack of familiarity with a *synthetic* writing inhibit the identification of the *relationships* between the various data, thus actually blocking the necessary premises for *recognition* - before utilisation - of the distributive property.

Protocol 2) also shows some separate operations, but the final writing opens a path towards the distributive property.

Solution 3) is the most evolved one, because it is the result of a *metacognitive* reading of the text, i.e. the problem solver does not merely "see" the problem information, but also acts upon it, connecting its different components.

Writing b) as hypothesised in these initial tests was not actually used. The hypothesis is that - as in 1) and 2) - the representations of the separate operations are seen by their authors as necessary intermediate stages, in order to arrive at the two partial re-

sults which will subsequently be added. "To add up two *operations*" – as children put it – appears incorrect, *anomalous*; a writing such as 1.5 x 7 + 1.5 x 3 keeps in a "pending" state as many as three operations, and this (confesses the first pupil) makes one anxious! In this case, a fundamental role would be played by the teachers having inducted the habit of considering as a number *all* its representations: from the *canonical* – 15.0 – to the *non canonical*, such as 1.5 x 7 + 1.5 x 3 ; 1.5 x (7 + 3) ; 1.5 x (7 + 3) , 10.5 + 4.5.

In this way, pupils would see the writing in a more reassuring light, as a sum of two numbers. Furthermore, it would be important for teachers to have habitually drawn the pupils' attention non only to the operational aspects - and the search for a result - thus encouraging procedural thinking, but also to rendering explicit the links between the various elements of problem information, and thus towards the relational aspects, which are far more important to the end of creating algebraic thinking in an arithmetical environment. Some teachers have nevertheless put forward the hypothesis that the presence of *different* objects (i.e. notebooks and pens) in the text may encourage the use of separate operations, given that it is customary to teach children in their first year of primary school that *"you cannot sum up apples and pears together, because you cannot sum up together things that are different"*. This kind of imprinting could well produce in some pupils (how many, we do not know) the tendency to use separate operations, actually considering the expression to be a mistake.

In conclusion, we can put forward the following hypotheses:

- The habit of making the children work in separate operations hinders or even totally inhibits the comprehension and the acquisition of the distributive property.
- Conversely, the habit of working with a synthetic representation of the solution of a problem (the expression) encourages the acquisition of control of the distributive property.

Brief general conclusions

The reported examples demonstrate how the behavioural patterns acquired by the pupils – as a result of teachers putting a greater emphasis on the procedural aspects of arithmetics – can hinder or even totally inhibit the pupils' approach to a task from an algebraic point of view. Therefore, the key element in the process of initiation of pre-algebraic thinking, as previously evidenced, is the teacher. This is also claimed by various scholars, even though some of them complain of rigid behaviour on the part of teachers even after a very long involvement (see the quotation by Blanton and Kaput, in Lee, 2001).

Our twenty years' experience allows us to assert that our interventions for and with teachers slowly modify their knowledge, behaviour and beliefs. In our intervention: a) we carry out a comparison among teachers of important research results; b) we conduct collective classroom discussions and so give teachers, *in vivo*, appropriate models of behaviour; c) we collectively analyse and reflect on teachers' attitudes, while we orchestrate discussions and select meaningful pupil protocols; d) we encourage teachers to actively participate in national conventions; etc.

In this way, teachers acquire a new style of managing classroom activities and get a new conception of their own profession (Malara and Iaderosa, ibid.; Malara and Navarra, 1998; Navarra, 2001). Of course, the depth of change in each teacher depends on her/his receptivity and willingness to re-appraise her/himself, and, moreover, on her/his being on the same wavelength as the leaders of innovation. What appears more problematic is the relapse of these experimentations at large in all schools, which are in many cases a world apart from any movement of renewal.

References

Ainley J., 1999, Doing algebra type stuff: emergent algebra in the primary school, *proc. PME 23*, v. 2, 9-16

Arcavi A.: 1994, Symbol sense: informal sense-making in formal mathematics, *For the Learning of Mathematics*, vol. 14, n.3, 24-35

Blanton, M., Kaput, J.: 2002, Design principles for tasks that support algebraic thinking in elementary school classroom, proc. PME 26, v. 2, 105-112

Brown L., Coles A., 1999, Needing to use Algebras – A case Study, *proc. PME 23*, v. 2, 153-160

Brito Lima A.P., Da Rocha Falcão J.T.: 1997, Early development of algebraic representation among 6-13 year-old children: the importance of didactic contract, *proc PME 21*, v. 2, 201-208

Carpenter T., Franke M.L.: 2001, Developing algebraic reasoning in the elementary school: generalization and proof, *Proc. 12 ICMI Study 'The future of the teaching & learning of Algebra'*, Melbourne, vol. 1: 155-162

Carraher D., Brizuela B., Earnest D.: 2001, The reification of additive difference in early algebra: viva la difference, *Proc. 12 ICMI Study, 'The future of the teaching and learning of Algebra'*, Melbourne, v. 1: 163-170

Chick,E., Stacey, K, Vincent, Jl., Vincent, Jn. (eds): 2001, *Proc. 12th ICMI Study 'The future of the teaching and learning of Algebra'*, Univ. Melbourne, Australia

Da Rocha Falcão, J.T.. Brito Lima, A.P., De Araújo, C. R., Lins Lessa, M.M., Osório, M.O: 2000, A didactic sequence for the introduction of algebraic activity in early elementary school, *proc. PME 24*, vol.2, 209-217

Doblaev L. P., 1969, Thought processes in setting up equations, in Kilpatrick J. Wirszup I. (eds) *Soviet Studies in the Psychology of Learning and Teaching Mathematics*, vol. III, National Council of Teachers of Mathematics, University of Chicago press, Chicago, 103-183

Even, R., Tirosh, D.: 2002, Teacher knowledge and understanding of students' mathematical learning,, in English, L. (ed) *Handbook of International Research in Mathematics Education*, LEA, NJ, 219-240

Jaworski, B.: 1998, Mathematics Teacher Research: Process, Practice and the Development of Teaching, *Journal of Mathematics Teacher Education*, n.1, 3-31

Kaput J., Blanton M.: 2001, Algebrafying the Elementary Mathematics Experience: Part 1: Transforming Task Structures, part II Transforming practice on a district-wide scale, *Proc. 12th ICMI Study 'The future of the teaching and learning of Algebra'*, Univ. Melbourne, vol.1, 344-353, 87-95

Kieran K.: 1992, The Learning and Teaching of School Algebra, in Grouws D.A. (edit.), *Handbook of Research on Mathematics Teaching and Learning*, Macmillan, NY, 390-419

Lee L.: 2001, Early algebra - but which algebra?, *Proc. 12th ICMI Study 'The future of the teaching and learning of Algebra'*, Univ. Melbourne, vol. 1, 392-399

Linchevski L.: 1995, Algebra with numbers and arithmetic with letters: a definition of pre-algebra, *Journal of Mathematical Behaviour*, vol. 14: 113 – 120

Malara, N. A.: 1999, Teaching and Learning of Algebra in Compulsory School: Questions and Results on a Long Term Research, in Rogerson, A. (ed), *Proc. Int. Conf. on Maths Education into 21st Century*, vol. 2, 68-78

Malara N.A., Navarra G.: 1998, Role of the teacher in promoting interaction among pupils and metacognition through problem solving abilities , *proc. CIEAEM 49*, 203-211

Malara N.A., Navarra G.: 2001, "Brioshi" and other mediation tools employed in a teaching of arithmetic with the aim of approaching algebra as a language, proc. 12th*ICMI Study on Algebra*, vol. 2, 412-419

Malara, N.A., Navarra, G.: 2002, ArAl: a Project for an early approach to algebraic thinking, *proc. Int. Conf. 'The Humanistic Renaissance in mathematics Education'*, Palermo, Italy, 228-233

Malara, N.A., Navarra, G., Giacomin, A., Iaderosa, R.: 2002, *The ArAl project: routes in arithmetic to promote prealgebraic thin*king, Pitagora, Bologna (in press), electronic version to be found on www.bdp.it

Menzel, B.: 2001, Language conceptions of algebra are idiosyncratic, in Chick, H. & al. (eds) 2001, *Proc. 12th ICMI Study 'The future of the teaching and learning of Algebra'*, Univ. Melbourne, Australia, 446-453

Savadosky P.: 1999, Arithmetic and algebraic practices: possible bridge between them, vol. 4, 145-152